The Dynamics of International Trade
with Variable Marginal Impatience

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Abstract

This paper presents a dynamic Heckscher–Ohlin–Samuelson (H–O–S) model with variable marginal impatience (of preference) in a small country economy. In the case of the increasing marginal impatience, the steady state with incomplete specialization exhibits uniqueness and saddle-point stability. On the other hand, in the case of the decreasing marginal impatience, contrary to the closed one-sector economy, the steady state with incomplete specialization does not exhibit uniqueness and saddle-point stability. We derive the trade pattern properties in the steady state.

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Keywords: Dynamic Heckscher–Ohlin–Samuelson model; Marginal impatience; Specialization and trade pattern

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1 Introduction

This paper presents a dynamic small country model of international trade with variable marginal impatience. Especially, we analyze the properties of dynamic system, that is the existence, the uniqueness, and the stability, and international trade pattern among the rest of world.

A large number of optimal growth theories assumed the time-separable utility function because of mathematical convenience. Most of empirical analyses show, however, that the time-separable utility function is not supported. The dynamic model with endogenous time preference was introduced by Uzawa (1968), and there has been an increasing use of in the growth theory. Epstein and Hynes (1983) examined some well-known dynamic models using a particular specification of the recursive utility. The utility function introduced by Uzawa (1968) was extended and clarified by Epstein (1987) and Obstfeld (1990).

As is well known, a problematic characteristic of the standard small open macroeconomic model with capital movement internationally is the fact that the existence of equilibrium requires the rate of time preference to be equal to the exogenously given world interest rate. The endogenous time preference framework, developed by Obstfeld (1981), Shi (1994), Karayalcin (1994, 1995), and Ikeda (2001), avoids these problems. The procedures adopted by the above-mentioned authors have significantly contributed to our model which assumes the small open economy model without international capital movement. This modification yields saddle-path dynamics for consumption, physical capital, and its shadow price.

On the other hand, the two-sector international trade model has a long tradition. “Heckscher–Ohlin–Samuelson (H–O–S) model” is the static international trade model in the neoclassical framework. The framework is characterized by constant returns to scale and perfect competition. The Rybczynski therem and the Stolper–Samuelson theorem are the important properties of the framework, which are applied to a variety of economic theories. This two-sector framework was introduced into the closed growth theory: the neoclassical growth model by Uzawa (1961,1963) and the optimal growth model by Uzawa (1964), Srinivasan (1964). Moreover, Lucas (1988), Mino (1996), and Bond, Wang, and Yip (1996) extended two-sector growth model to the endogenous growth theory.

The dynamic H–O–S model originates in Oniki and Uzawa (1965). Oniki and Uzawa (1965) assumed two country, two goods, two factors, and the neoclassical growth framework, while
Stiglitz (1970), Manning (1981), Baxter (1992), and Chen (1992) assumed the optimal growth framework. These optimal growth models, however, lead to unrealistic results as follows:

- There is a crucial price with long-run diversified production, in other word, this economy is likely to specialize completely in one good in the steady state.
- The long-run equilibria with diversified production is not unique.
- There is no transition dynamics for the long-run equilibria with diversified production.

These properties contrast the static H–O–S model, in which the slight difference in the conditions like technologies, preferences, or economic policies among countries does not cause specialization. Baxter (1992, p.714) argued for these properties that “The predictions of this model regarding patterns of specialization and trade are markedly different from those of HOS model but are very much in the spirit of the traditional Ricardian model. That is, the equilibrium pattern of specialization and trade depends on comparative advantage and there is a corresponding presumption of specialization.” The determinant of long-run comparative advantage in the dynamic H–O–S model mentioned above is the countries’ exogenous time preference rate. The incomplete specialization holds only if time preference rates, which are exogenous and constant, are strictly identical between the economy and the rest of world. This property in dynamic H–O–S model is not apparently satisfactory.

The purpose of this paper is to construct a dynamic international trade model in which leads more realistic results. To achieve this purpose, we introduce “the marginal impatience (of preference)” into the small country dynamic international trade model. Especially, we analyze the cases of (i) the constant marginal impatience (time-separable utility), (ii) the increase marginal impatience (Uzawa–Epstein utility), and (iii) the decreasing marginal impatience (Das (2003), Chang (2004)), respectively. Das (2003) and Chang (2004) proved that contrary to the general belief, a negative relationship between the discount rate and the consumption of consumer does not necessarily result instability of the dynamic system in the closed one-sector economy. We apply the formulation to our analysis.

The analysis presented in this paper reveals the importance of two features. In the case of the increasing marginal impatience, the steady state with incomplete specialization exhibits uniqueness and saddle-point stability. On the other hand, in the case of the decreasing marginal
impatience, contrary to the closed one-sector economy, the steady state with incomplete specialization does not exhibit uniqueness and saddle-point stability.

The rest of the paper is organized as follows: Section 2 presents the model and derives the optimality conditions. In section 3, we characterize the steady state an equilibrium dynamics. In section 4, we derive the trade pattern properties. In section 5, we consider the properties in the case of the decreasing marginal impatience. Section 6 concludes the paper.

2 Model

This section specifies a continuous-time Ramsey model of a small open economy with perfective competitive world. The economy consists of many infinitely lived identical agents (households and firms). There are two sectors in this economy that produce a pure consumption good (good 1) and a pure investment good (good 2). We choose good 2 as the numeraire. We normalize the initial number of people to unity and simplify by assuming that population grows at a constant, exogenous rate, \( n \in [0, \infty) \). Following the tradition of the dynamic international trade theory, we assume that while the two goods are tradable, the production factors are not mobile internationally.

2.1 Production

The output of good 1 and good 2 are generated by using capital and labor. Technology is specified by two production functions which are homogenous of degree one in both factors, and is stationary over time. The production function of each producer can be written in terms of outputs per worker as follows:

\[
\begin{align*}
    l_1(t)f_1(k_1(t)), \\
    l_2(t)f_2(k_2(t)),
\end{align*}
\]

where \( k_i(t) \) is the capital-labor ratio in sector \( i \), and \( l_i(t) \in [0, 1] \) is the proportion of the labor employed in sector \( i \) at time \( t \in [0, \infty), i = 1, 2 \), respectively. We impose the following assumptions:
Assumption 1 (the production function). The production function \( f_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is twice continuously differentiable, with

(i) \( f_i' (k_i(t)) > 0, \quad f_i'' (k_i(t)) < 0, \quad \forall k_i(t) > 0, \quad i = 1, 2, \)

(ii) \( \lim_{k_i(t)\downarrow 0} f_i' (k_i(t)) = \infty, \quad \lim_{k_i(t)\rightarrow \infty} f_i' (k_i(t)) = 0, \quad f_i (0) = 0, \quad i = 1, 2. \)

Assumption 2 (no factor intensity reversal). There are no factor intensity reversals, that is, either \( k_2(t) > k_1(t), \quad \forall p(t) \in (0, \infty), \) or \( k_2(t) < k_1(t), \quad \forall p(t) \in (0, \infty), \) where \( p(t) \) is the price of the good 1 in terms of good 2.

With full mobility of factors across sectors, the resource constraints require that

\[
l_1(t)k_1(t) + l_2(t)k_2(t) \leq k(t), \quad (2a)
\]

\[
l_1(t) + l_2(t) \leq 1, \quad (2b)
\]

where \( k(t) \in [0, \infty) \) is the aggregate capital-labor ratio.

The per capita GDP function can be defined as:

\[
g(k(t), p(t)) \equiv \max_{y(t), \{k_i(t)\}, \{l_i(t)\}} \left\{ p(t) \cdot y(t) : y_i(t) \leq l_i(t)f_i(k_i(t)), \sum_i l_i(t)k_i(t) \leq k(t), \right. \\
\left. \sum_i l_i(t) \leq 1, \quad y_i(t) \geq 0, \quad k_i(t) \geq 0, \quad l_i(t) \geq 0, \quad i = 1, 2 \right\},
\]

(P)

where \( y_i(t) \) is the per capita output in the sector \( i, \) \( p(t) \equiv (p(t), 1), \) and \( y(t) \equiv (y_1(t), y_2(t)). \)

In the optimality, it is satisfied \( y_i(t) = l_i(t)f_i(k_i(t)), \) \( \sum_i l_i(t) = 1, \) \( \sum_i l_i(t)k_i(t) = k(t), \quad i = 1, 2. \)

The GDP function has the properties

\[
\lim_{k(t)\downarrow 0} g_k (k(t), p(t)) = \infty, \quad \lim_{k(t)\rightarrow \infty} g_k (k(t), p(t)) = 0, \quad g (0, p(t)) = 0,
\]

and is concave in \( k(t) \) and convex in \( p(t) \) as in Figure 1 which are in the case of \( (k(t), p(t)) = (k_0, p_0)^1. \) These figures also show that it is the upper envelope of \( p(t)f_1 (k_1(t)) \) and \( f_2 (k_2(t)). \)

[Figure 1 about here.]

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1For details of Figure 1, see Deardorff (1974).
If both goods are produced, then

\[ r(t) = p(t)f'_1(k_1(t)) = f'_2(k_2(t)), \]
\[ w(t) = p(t) \left[ f_1(k_1(t)) - k_1(t)f'_1(k_1(t)) \right] = \left[ f_2(k_2(t)) - k_2(t)f'_2(k_2(t)) \right], \]

where \( r(t) \) is the rental rate of capital and \( w(t) \) is the wage rate in terms of good 2, respectively. Therefore, as long as both goods are produced, it follows from equations above that \( k_i(p(t)) \) in Figure 1 is

\[ k_i(t) = k_i(p(t)), \quad \frac{dk_i(p(t))}{dp(t)} \begin{cases} < 0, & \text{if } k_1(p(t)) > k_2(p(t)) \\
> 0, & \text{if } k_1(p(t)) < k_2(p(t)) \end{cases}, \quad i = 1, 2, \forall p(t) \in (0, \infty). \quad (3) \]

The labor allocations \( l_i(t) \) is then obtained from Eq. (2),

\[ l_1(t) \equiv l_1(k(t), p(t)) = \frac{k(t) - k_2(p(t))}{k_1(p(t)) - k_2(p(t))}, \]
\[ l_2(t) \equiv l_2(k(t), p(t)) = \frac{k_1(p(t)) - k(t)}{k_1(p(t)) - k_2(p(t))}. \]

Then, given the capital-labor ratio \( k(t) \), we obtain the pattern of the specialization as follows:

\[ \begin{cases} 0 < k(t) \leq k_i(p(t)), & \text{specialized to the good } i, \\
k_i(p(t)) < k(t) < k_j(p(t)), & \text{incompletely specialized,} \\
k_j(p(t)) \leq k(t) < \infty, & \text{specialized to the good } j, \end{cases} \]

for \( k_i(p(t)) < k_j(p(t)), \ i, j = 1, 2, \ i \neq j, \ \text{and} \ \forall p(t) \in (0, \infty). \) Furthermore, by using Eq.(2a) and (3), the pattern of the specialization is reduced to

\[ \begin{cases} 0 < p(t) \leq p_2(k(t)), & \text{specialized to the good } 2, \\
p_2(k(t)) < p(t) < p_1(k(t)), & \text{incompletely specialized,} \\
p_1(k(t)) \leq p(t) < \infty, & \text{specialized to the good } 1, \end{cases} \]

where \( p_i(k(t)) \equiv k^{-1}_i(k(t)), \ i = 1, 2, \ \text{for} \ \forall k(t) \in [0, \infty), \ \text{regardless of} \ k_1(p(t)) \geq k_2(p(t)). \)
Finally, we can define the set of diversified production with respect to \((k(t), p(t))\) as follows:

\[
\Gamma \equiv \{(k(t), p(t)) \in \mathbb{R}_{++} \times \mathbb{R}_{++} : k(t) \in (k_i (p(t)), k_j (p(t))), k_i (p(t)) < k_j (p(t)), i, j = 1, 2, i \neq j\}.
\]

This set is illustrated by the shaded region between schedules \(k_1 (p(t))\) and \(k_2 (p(t))\) in \((k(t), p(t))\) plane of Figure 1. This economy produces both goods only if the relative price \(p(t)\) and the capital-labor ratio \(k(t)\) belong to the region.

By using the envelope theorem, we have the following properties for the GDP function:

\[
y_1(t) \equiv y_1 (k(t), p(t)) = g_p (k(t), p(t)) , \quad (4a)
\]

\[
y_2(t) \equiv y_2 (k(t), p(t)) = g (k(t), p(t)) - p(t)g_p (k(t), p(t)) , \quad (4b)
\]

\[
r(t) \equiv r (k(t), p(t)) = g_k (k(t), p(t)) , \quad (4c)
\]

\[
w(t) \equiv w (k(t), p(t)) = g (k(t), p(t)) - k(t)g_k (k(t), p(t)) . \quad (4d)
\]

\[
p(t)y_1(t) + y_2(t) = r(t)k(t) + w(t) = g (k(t), p(t)) , \quad (4e)
\]

Let \(\hat{r}(t)\) be the rental rate of capital under incomplete specialization, then it is the slope of tangent AB in Figure 1 and has the properties

\[
\hat{r}(t) \equiv \hat{r} (p(t)) = p(t)f'_1 (k_1 (p(t))) = f'_2 (k_2 (p(t))) , \quad (4c')
\]

\[
\left\{ \begin{array}{ll}
\hat{r}' (p(t)) > 0, & \text{if } k_1 (p(t)) > k_2 (p(t)) \\
\hat{r}' (p(t)) < 0, & \text{if } k_1 (p(t)) < k_2 (p(t)) \end{array} \right. , \forall p(t) \in (p_2 (k(t)), p_1 (k(t))).
\]

Furthermore, let \(\hat{w}(t)\) be the wage rate under incomplete specialization, then it is the vertical intercept of tangent AB in Figure 1 and has the properties

\[
\hat{w}(t) \equiv \hat{w} (p(t)) = p(t) \left[ f_1 (k_1 (p(t))) - k_1 (p(t)) f'_1 (k_1 (p(t))) \right] = f_2 (k_2 (p(t))) - k_2 (p(t)) f'_2 (k_2 (p(t))) , \quad (4d')
\]

\[
\left\{ \begin{array}{ll}
\hat{w}' (p(t)) < 0, & \text{if } k_1 (p(t)) > k_2 (p(t)) \\
\hat{w}' (p(t)) > 0, & \text{if } k_1 (p(t)) > k_2 (p(t)) \end{array} \right. , \forall p(t) \in (p_2 (k(t)), p_1 (k(t))).
By using $\dot{r}(p(t))$, $\dot{w}(p(t))$, and (4), we have

\[ y_1(k(t), p(t)) = \dot{r}'(p(t)) k(t) + \dot{w}'(p(t)), \quad (4a') \]
\[ y_2(k(t), p(t)) = [\dot{r}(p(t)) - p(t)\dot{r}'(p(t))] k(t) + [\dot{w}(p(t)) - p(t)\dot{w}'(p(t))] \]
\[ p(t)y_1(k(t), p(t)) + y_2(k(t), p(t)) = \dot{r}(p(t)) k(t) + \dot{w}(p(t)), \quad (4e') \]

under incomplete specialization. We can summarize these results as follows:

\[
\begin{align*}
  &g_k(k(t), p(t)) \geq 0, \quad \text{with equality when } k(t) \to \infty, \\
  &g_{kk}(k(t), p(t)) \leq 0, \quad \text{with equality when } p(t) \in (p_2(k(t)), p_1(k(t))), \\
  &g_p(k(t), p(t)) \geq 0, \quad \text{with equality when } p(t) \in (0, p_2(k(t))), \\
  &g_{pp}(k(t), p(t)) \geq 0, \quad \text{with inequality when } p(t) \in (p_2(k(t)), p_1(k(t))), \\
  &g_{kp}(k(t), p(t)) = g_{pk}(k(t), p(t)).
\end{align*}
\]

Lemmas 1 and 2 provide the properties of $g_{kp}(k(t), p(t))$ under incomplete specialization:

**Lemma 1** (Rybczynski). Suppose that $y(t) \gg 0$, Assumption 1, and 2 is satisfied.

(i) If $k_2(p(t)) < k_1(p(t))$, $\forall p(t) \in (p_2(k(t)), p_1(k(t)))$, then

\[
\frac{\partial y_1(t)}{\partial k(t)} = \frac{\partial y_2(t)}{\partial k(t)} = g_k(k(t), p(t)) - p(t)g_{pk}(k(t), p(t)) < 0.
\]

(ii) If $k_2(p(t)) > k_1(p(t))$, $\forall p(t) \in (p_2(k(t)), p_1(k(t)))$, then

\[
\frac{\partial y_1(t)}{\partial k(t)} = \frac{\partial y_2(t)}{\partial k(t)} = g_k(k(t), p(t)) - p(t)g_{pk}(k(t), p(t)) < 0.
\]

**Lemma 2** (Stolper–Samuelson). Suppose that $y(t) \gg 0$, Assumption 1, and 2 is satisfied.

(i) If $k_2(p(t)) < k_1(p(t))$, $\forall p(t) \in (p_2(k(t)), p_1(k(t)))$, then

\[
\frac{dr(t)/dr(t)}{p(t)} = \frac{p(t)g_{kp}(k(t), p(t))}{g_k(k(t), p(t))} > 1,
\]

\[
\frac{dw(t)/dp(t)}{p(t)} = g_p(k(t), p(t)) - k(t)g_{kp}(k(t), p(t)) < 0.
\]

(ii) If $k_2(p(t)) > k_1(p(t))$, $\forall p(t) \in (p_2(k(t)), p_1(k(t)))$, then

\[
\frac{dr(t)/dr(t)}{p(t)} = \frac{p(t)g_{kp}(k(t), p(t))}{g_k(k(t), p(t))} > 1,
\]

\[
\frac{dw(t)/dp(t)}{p(t)} = g_p(k(t), p(t)) - k(t)g_{kp}(k(t), p(t)) < 0.
\]

---

\[
\frac{dr(t)}{dp(t)} = g_{kp}(k(t), p(t)) < 0, \\
\frac{dw(t)/dp(t)}{w(t)/p(t)} = \frac{p(t) [g_p(k(t), p(t)) - k(t)g_{kp}(k(t), p(t))]}{g(k(t), p(t)) - k(t)g_k(k(t), p(t))} > 1.
\]

As is well known, these lemmas show the magnification effects (see Jones (1965)). On comparing Lemma 1 with 2, we observe the fact that these properties are equivalent, that is the reciprocity relation (See Samuelson (1953)). Furthermore, we obtain Lemma 3 as the properties of \( g_{kp}(k(t), p(t)) \) under complete specialization in good 1 or good 2, because \( g(k(t), p(t)) = f_2(k(t)), \forall p(t) \in (0, p_2(k(t))] \) and \( g(k(t), p(t)) = p(t)f_1(k(t)), \forall p(t) \in [p_1(k(t)), \infty). \)

**Lemma 3.** Suppose that Assumption 1 and 2 is satisfied.

(i) \( \forall p(t) \in (0, p_2(k(t)]), g_{kp}(k(t), p(t)) = \frac{\partial^2 f_2(k(t))}{\partial k(t) \partial p(t)} = 0. \)

(ii) \( \forall p(t) \in [p_1(k(t)), \infty), g_{kp}(k(t), p(t)) = \frac{\partial}{\partial p(t)} \left[ \frac{\partial p(t)f_1(k(t))}{\partial k(t)} \right] = \frac{\partial^2 p(t)f_1(k(t))}{\partial k(t) \partial p(t)} = f_1'(k(t)) > 0. \)

### 2.2 Household

Consider an infinitely lived consumer who maximizes the lifetime utility by choosing the time profile of consumption \( C \equiv \{c(t) : t \geq 0\} \). It permits (\( C_T, \tau C \)) for all \( T \in (0, \infty) \), where \( C_T \) and \( \tau C \) are the programs up to time \( T \) and after \( T \), respectively. The intertemporal utility function is defined over such programs as follows:

\[
U(C) \equiv \int_0^\infty u(c(t)) \exp \left[ - \int_0^t \{\delta(c(\tau)) - n\} d\tau \right] dt,
\]

where \( u(c(t)) \) and \( \delta(c(t)) \) are the “felicity” or instantaneous utility function and the subjective discount function, respectively. The following assumptions characterize function \( u(c(t)). \)

**Assumption 3** (the felicity function). *The felicity function \( u : \mathbb{R}_+ \to \mathbb{R} \) is real valued, bounded above, \( \lim_{c(t) \to 0} u'(c(t)) = \infty \), and twice continuously differentiable with \( u'(c(t)) > 0 \) and \( u''(c(t)) \leq 0, \forall c(t) \in [0, \infty). \)

Next, we define the concepts for the marginal impatience as follows:

**Definition 1** (the constant marginal impatience). *It is said to be “constant marginal impatience” if \( \delta'(c(t)) = 0, \forall c(t) \in [0, \infty). \)
Definition 2 (the increasing marginal impatience). It is said to be "increasing marginal impatience" if \( \delta'(c(t)) > 0 \), \( \forall c(t) \in [0, \infty) \).

Definition 3 (the decreasing marginal impatience). It is said to be "decreasing marginal impatience" if \( \delta'(c(t)) < 0 \), \( \forall c(t) \in [0, \infty) \).

If people have constant marginal impatience, \( \delta(c(t)) \) is constant which yields the conventional time-separable utility function. On the other hand, if they have increasing or decreasing marginal impatience, the time preference rate is determined endogenously by the present and future consumption. We characterize the case of \( \delta'(c(t)) = 0 \) and \( \delta'(c(t)) > 0 \) from this section to Section 4, and consider the case of \( \delta'(c(t)) < 0 \) in Section 5. Then, the function \( \delta(c(t)) \) is assumed as follows:

Assumption 4 (the subjective discount rate function). The subjective discount rate function \( \delta : \mathbb{R}_+ \to \mathbb{R}_{++} \) is real valued, bounded above, \( \delta(c(t)) > n \), and twice continuously differentiable with \( \delta'(c(t)) \geq 0 \) and \( \delta''(c(t)) \leq 0 \), \( \forall c(t) \in [0, \infty) \).

Remark 1. The assumption of increasing marginal impatience is the most controversial in the theory of endogenous time preference rate \(^3\). Lucas and Stokey (1984) assumed it to ensure local stability of steady state. Epstein (1987, pp. 73-74) justified it from two reasons besides above: (i) \( \delta'(c(t)) > 0 \) is implied by the hypothesis \( \partial \rho(c(T), U(TC), n)/\partial U(TC) > 0 \), where function \( \rho(c(T), U(TC), n) \) is the time preference rate and \( U(TC) \) represents the lifetime utility \( U \) from the consumption stream after time \( T \), and this hypothesis is plausible on introspective grounds, and (ii) the extension of the increasing marginal impatience to a stochastic framework is desirable and the specification of function \( U(C) \) as a "von Neumann–Morgenstern index" is natural.

Note that the utility function adopted here is generalization of the Uzawa (1968) functional and the Epstein’s recursive utility framework (see Epstein (1987, Lemma 1)) as well as the time-separable utility function \(^4\).

\(^3\)See also Blanchard and Fischer (1989) and Obstfeld (1990) for further details.

\(^4\)Uzawa functional satisfies properties that \( u(c(t)) \geq 0 \), \( u(0) = 0 \), \( u'(c(t)) > 0 \), \( u''(c(t)) < 0 \), \( \delta(c(t)) \equiv \delta(u(c(t))) > 0 \), \( \delta'(u(c(t))) > 0 \), \( \delta''(u(c(t))) > 0 \), \( \delta'(u(c(t))) - u(c(t)) \delta''(u(c(t))) > 0 \), and \( \delta''(u(c(t))) u'(c(t))^2 \leq 0 , \forall c(t) \in [0, \infty) \) (See Obstfeld (1990)). On the other hand, Epstein (1987) assumed that \( -\infty < \inf u(c(t)) < \sup u(c(t)) < 0 \), \( u'(c(t)) > 0 \), \( \log \left(-u(c(t))\right) \) is convex, \( \inf \delta(c(t)) > 0 \), \( \delta'(c(t)) > 0 \), and \( \delta''(c(t)) \leq 0 \), \( \forall c(t) \in [0, \infty) \). Note that we rule out the Epstein and Hynes (1983) functional \( u(c(t)) = -1 \), \( \forall c(t) \in [0, \infty) \), to analyze the time-separable utility function as a special case.
To characterize variable marginal impatience more clearly, we follow Drugeon (1996) in introducing the subjective discount function $\delta(c(t))$ as

$$\delta(c(t)) \equiv \delta_0 + \theta(c(t)), \quad \delta_0 > n, \quad (6)$$

where the function $\theta(c(t))$ requires that $\theta(0) = 0$, $\theta'(c(t)) = \delta'(c(t)) \geq 0$ and $\theta''(c(t)) = \delta''(c(t)) \leq 0$, $\forall c(t) \in [0,\infty)$, in order to satisfy Assumption 4. If $\theta(c(t)) = 0$ we have time-separable utility function, and if $\theta'(c(t)) > 0$, $\theta''(c(t)) \leq 0$ we have the increasing marginal impatience, respectively.

[Figure 2 about here.]

The marginal utility of $c(T)$ is defined in terms of the “Volterra derivative” as follows $^5$:

$$\frac{\partial U(C)}{\partial c(T)} = [u'(c(T)) - \delta'(c(T))U(TC)] \exp\left[-\int_0^T \{\delta(c(t)) - n\}d\tau\right], \quad (7)$$

where

$$U(TC) \equiv \int_0^\infty u(c(t)) \exp\left[-\int_T^{t} \{\delta(c(\tau)) - n\}d\tau\right]dt.$$

Following Epstein and Hynes (1983), we next define the rate of time preference, $\rho(t)$, as the rate of decrease in marginal utility of consumption along a locally constant path:

$$\rho(T) \equiv \rho(c(T), U(TC), n) = -\frac{d}{dT} \log \left. \frac{\partial U}{\partial c(T)} \right|_{c(T)=0}$$

$$= [\delta(c(T)) - n] - \frac{u(c(T)) - [\delta(c(T)) - n]U(TC)}{u'(c(T)) - \delta'(c(T))U(TC)} \delta'(c(T)), \quad (8)$$

which shows that $\rho(T)$ depends on current consumption $c(T)$ and $TC \equiv \{c(t) \mid t \geq T\}$. Thus, we can consider a rate of time preference function, $\rho(c(T), U(TC), n)$, in which there is no harm in setting $T = 0$. We can easily check that in the case of the time-separable preference $\delta'(c(T)) = 0$, $\rho(c(T), U(TC), n) = \delta(c(T)) - n = \delta_0 - n$.

**Assumption 5.** $\forall c(t) \in [0,\infty)$, $u(c(t))$ and $\delta(c(t))$ satisfies the following conditions:

(i) $u'(c(t)) - \delta'(c(t))U(TC) > 0$ and $\lim_{c(t) \to 0} [u'(c(t)) - \delta'(c(t))U(TC)] = \infty,$

---

$^5$See Wan (1970) and Ryder and Heal (1973) for details of Volterra derivative.
(ii) \( u'(c(t)) [\delta(c(t)) - n] - u(c(t)) \delta'(c(t)) > 0 \).

Assumption (5.i) ensures positive marginal utility and interior solution for the optimality, and Assumption (5.ii) ensures that time preference rate is positive, respectively. Note that \( u'(c(t)) \) is “marginal felicity”, which is not same as “marginal utility”, as defined by Eq. (7).

**Assumption 6.** \( u''(c(t)) - \delta''(c(t)) U(tC) < 0, \forall c(t) \in [0, \infty) \).

This assumption ensures that the Hamiltonian, which is specified below, is concave \( \forall c(t) \in [0, \infty) \).

Let us construct “the generating function” \( v \),

\[
v(c(t), U(tC), n) \equiv u(c(t)) - [\delta(c(t)) - n] U(tC),
\]

where \( v_c(c(t), U(tC)) > 0, v_{cc}(c(t), U(tC)) < 0 \) from Assumption 5 and 6.

If we define the cumulated “net” discount rate as \( \int_0^t \{\delta(c(\tau)) - n\} d\tau \) and

\[
\Delta(t) \equiv -\int_0^t \{\delta(c(\tau)) - n\} d\tau, \quad \Delta(0) = 0,
\]

then the individual’s utility function can be expressed as follows:

\[
\begin{cases}
U = \int_0^\infty u(c(t)) \exp \Delta(t) dt, \\
\dot{\Delta}(t) = -\{\delta(c(\tau)) - n\}, \quad \Delta(0) = 0.
\end{cases}
\]

(5’)

where a “dot” over a variable denotes the time derivative of the variable. The flow budget constraint for the household at time \( t \) is

\[
\dot{k}(t) = g(k(t), p(t)) - (\mu + n) k(t) - p(t) c(t),
\]

(11)

where \( \mu \in [0, \infty) \) denotes the rate of capital depreciation.
The consumer solves the following household’s maximizing problem:

\[
\begin{align*}
\max_C & \quad \int_0^\infty u(c(t)) \exp \Delta(t) dt, \\
\text{s.t.} & \quad \dot{k}(t) = g(k(t), p(t)) - (\mu + n) k(t) - p(t)c(t), \\
& \quad \dot{\Delta}(t) = -[\delta(c(t)) - n], \\
& \quad k(0) = k_0, \quad \Delta(0) = 0: \text{given.}
\end{align*}
\]  

(H)

The present-value Hamiltonian for problem (H) is

\[
\mathcal{H}(c(t), k(t), \Delta(t), \Lambda(t), \Phi(t); p(t)) = u(c(t)) \exp \Delta(t) + \Lambda(t) [g(k(t), p(t)) - (\mu + n) k(t) - p(t)c(t)] - \Phi(t) [\delta(c(t)) - n],
\]

where \(\Lambda(t)\) and \(\Phi(t)\) are the costate variables of \(k(t)\) and \(\Delta(t)\), respectively. The first-order conditions of this problem are given by

\[
\frac{\partial \mathcal{H}(t)}{\partial c(t)} = 0; \quad u'(c(t)) \exp \Delta(t) - p(t)\Lambda(t) - \Phi(t)\delta'(c(t)) = 0,
\]

\[
\frac{\partial \mathcal{H}(t)}{\partial k(t)} = -\dot{\Lambda}(t); \quad [g_k(k(t), p(t)) - (\mu + n)] \Lambda(t) = -\dot{\Lambda}(t),
\]

\[
\frac{\partial \mathcal{H}(t)}{\partial \Delta(t)} = -\Phi(t); \quad u(c(t)) \exp \Delta(t) = -\Phi(t),
\]

transversality condition: \(\lim_{t \to \infty} k(t)\Lambda(t) = \lim_{t \to \infty} \Delta(t)\Phi(t) = 0\).

Let \(\lambda(t) \equiv \Lambda(t) \exp[-\Delta(t)]\) and \(\phi(t) \equiv \Phi(t) \exp[-\Delta(t)]\) represent the current-value costate variables for \(k(t)\) and \(\Delta(t)\), respectively. First-order conditions of the problem are then

\[
p(t)\lambda(t) = u'(c(t)) - \phi(t)\delta'(c(t)), \tag{13a}
\]

\[
\dot{\lambda}(t) = [\delta(c(t)) - \{g_k(k(t), p(t)) - \mu\}] \lambda(t), \tag{13b}
\]

\[
\dot{\phi}(t) = -[u(c(t)) - \phi(t) \{\delta(c(t)) - n\}], \tag{13c}
\]

\[
\lim_{t \to \infty} k(t)\lambda(t) \exp \Delta(t) = \lim_{t \to \infty} \Delta(t)\phi(t) \exp \Delta(t) = 0. \tag{13d}
\]

Eq. (13a) implies that the marginal benefit of consumption must be equal to the respective marginal cost in terms of utility. Eqs. (13b) and (13c) are the Euler equations associated with
\( \lambda(t) \) and \( \phi(t) \), respectively. Eq. (13d) is the transversality condition \(^6\). On using (13c) and (13d), we obtain \(^7\)

\[
\phi(T) = \int_T^\infty u(c(t)) \exp \left[ -\int_T^t \{ \delta(c(t)) - n \} dt \right] dt = U(T_C).
\]

Therefore, \( \phi(T) \) represents the lifetime utility after time \( T \), and it is satisfied \( \lambda(T) > 0 \) from Assumption 5.

### 3 Dynamic System

This section specifies a complete dynamic system a small open economy and discusses the existence, uniqueness, and stability of it.

By the small-country assumption, the relative price is exogenously given at \( p \). The optimality conditions of the production sector and the household sector are composed of Eqs. (11) and (13), respectively. Noting that \( p\dot{\lambda} = v_{cc}(c, \phi, n) \dot{c} + \delta'(c) v(c, \phi, n) \), and using (13a) and (13c), the complete dynamic system with respect to \( c \), \( \phi \), and \( k \) is constituted by the differential equations

\(^6\) From \( \delta'(c(t)) \geq 0 \), we have

\[ \delta(c(t)) - n = \theta(c(t)) + \delta_0 - n \geq \delta_0 - n > 0. \]

Thus, \( \forall t \in (0, \infty) \), we obtain

\[
\int_0^t \{ \delta(c(t)) - n \} dt \geq \int_0^t \{ \delta_0 - n \} dt = (\delta_0 - n) t > 0
\]

\[ \Rightarrow \Delta(t) = -\int_0^t \{ \delta(c(t)) - n \} dt \leq - (\delta_0 - n) t < 0 \]

\[ \Rightarrow 0 \leq \lim_{t \to \infty} \exp \Delta(t) \leq \lim_{t \to \infty} \exp [- (\delta_0 - n) t] = 0 \]

\[ \Rightarrow \lim_{t \to \infty} \exp \Delta(t) = 0. \]

\(^7\) From footnote 6, \( \Delta(t) < 0, \forall t \in (0, \infty) \). Therefore,

\[ \lim_{t \to \infty} \Delta(t) \Phi(t) = 0 \Rightarrow \lim_{t \to \infty} \Phi(t) = \lim_{t \to \infty} \phi(t) \exp \Delta(t) = 0. \]

From (13c) and \( \lim_{t \to \infty} \phi(t) \exp \Delta(t) = 0 \), we derive (14).
as follows\(^8\):

\[
\dot{c} = \sigma(c, \phi) \left[ \{g_k(k, p) - (\mu + n)\} - \rho(c, \phi, n) \right] c,
\]

\[
\dot{k} = g(k, p) - (\mu + n) k - pc,
\]

\[
\dot{\phi} = -v(c, \phi, n) \equiv - [u(c) - \phi \{\delta(c) - n\}],
\]

where

\[
\sigma(c, \phi) \equiv - \frac{u'(c) - \phi\delta'(c)}{[u''(c) - \phi\delta''(c)]} c = - \frac{v_c(c, \phi, n)}{v_{cc}(c, \phi, n)} c > 0.
\]

### 3.1 Existence and Uniqueness of the Steady State

The steady state of this system is realized when all variables in (15) stay constant over time. The steady state \((\bar{c}, \bar{k}, \bar{\phi})\) is characterized by the set of conditions

\[
\rho(\bar{c}, \bar{\phi}, n) = \delta(\bar{c}) - n = g_k(\bar{k}, p) - (\mu + n), \quad (16a)
\]

\[
g(\bar{k}, p) = (\mu + n) \bar{k} + p\bar{c}, \quad (16b)
\]

\[
\bar{\phi} = \frac{u(\bar{c})}{\delta(\bar{c}) - n}, \quad (16c)
\]

where \(\delta(\bar{c}) = \delta_0 + \theta(\bar{c})\), and therefore we can express \(\bar{c}, \bar{k}, \) and \(\bar{\phi}\) as functions \(\bar{c} \equiv c(p, n, \delta_0), \\bar{k} \equiv k(p, n, \delta_0), \) and \(\bar{\phi} \equiv \phi(p, n, \delta_0), \) respectively. Using (16b), the steady state value of per capita consumption, \(\bar{c}\), is given by

\[
\bar{c} = \frac{g(\bar{k}, p) - (\mu + n) \bar{k}}{p}. \quad (16b')
\]

Then we can derive the following reduced-form equation characterizing the steady state in terms of a single variable \(\bar{k}\):

\[
\delta \left( \frac{g(\bar{k}, p) - (\mu + n) \bar{k}}{p} \right) = g_k(\bar{k}, p) - \mu. \quad (17)
\]

Once we find a solution for the equilibrium capital stock, \(\bar{k}\), from Eq.(17), the corresponding values of consumption, \(\bar{c}\), and lifetime utility, \(\bar{\phi}\), can be derived from (16b') and (16c), respectively.

\(^8\)For simplicity, we henceforth omit the time arguments excepted where needed for clarity.
In order to analyze the **existence** and **uniqueness** of steady state, we define these two equations\(^9\):

\[
\begin{align*}
\text{RHS} (k, p) & \equiv g_k (k, p) - \mu, \\
\text{LHS} (k, p, n) & \equiv \delta \left( \frac{g (k, p) - (\mu + n) k}{p} \right). 
\end{align*}
\] (18a) (18b)

Note that as shown of Figure 3, we can confine the analysis to the region in which \( k \in [0, k_c (p, n)] \), where \( k_c = k_c (p, n) \) is the capital stock level at which output per head is just sufficient to replace depreciation per head, that is, \( k_c (p, n) \equiv \{ k : g (k, p) = (\mu + n) k, \ k > 0 \} \), because of the interior solution about \( c \).

[Figure 3 about here.]

The properties of function \( \text{RHS} (k, p) \) is as follows:

\[
\begin{align*}
\lim_{k \downarrow 0} \text{RHS} (k, p) &= \infty, \\
\text{RHS} (k_g (p, n), p) &= n, \\
\text{RHS} (k, p) &= \hat{r} (p) - \mu, \quad k \in (k_i (p), k_j (p)), \\
\text{RHS} (k_i (p), p) = \text{RHS} (k_j (p), p) &= \hat{r} (p) - \mu, \\
\frac{\partial \text{RHS} (k, p)}{\partial k} &= g_{kk} (k, p) \begin{cases} 
< 0, & \text{if } k \in [0, k_i (p)] \cup [k_j (p), \infty) \\
= 0, & \text{if } k \in (k_i (p), k_j (p)) 
\end{cases}, \quad k_i (p(t)) < k_j (p(t)), \ i, j = 1, 2, i \neq j,
\end{align*}
\]

where \( k_g = k_g (p, n) \) is golden rule capital stock, which satisfies \( g_k (k_g, p) = \mu + n \) and \( k_g (p, n) \in \).

---

\(^9\)For the analysis in this section, see also Das (2003).
Moreover, the properties of function \( \text{LHS}(k, p, n) \) is as follows:

\[
\lim_{k \downarrow 0} \text{LHS}(k, p, n) = \text{LHS}(k_c(p, n), p, n) = \delta_0
\]

\[
\frac{\partial \text{LHS}(k, p, n)}{\partial k} = \frac{\delta'}{p} [g_k(k, p) - (\mu + n)] \begin{cases} 
= 0, & \text{if } \delta'(c) = 0, \\
> 0, & \text{if } \delta'(c) > 0 \text{ and } k \in [0, k_g(p, n)), \\
= 0, & \text{if } \delta'(c) > 0 \text{ and } k = k_g(p, n), \\
< 0, & \text{if } \delta'(c) > 0 \text{ and } k \in (k_g(p, n), k_c(p, n)). 
\end{cases}
\]

\[
\frac{\partial^2 \text{LHS}(k, p, n)}{\partial k^2} = \frac{\delta''}{p^2} [g_k(k, p) - (\mu + n)]^2 + \frac{\delta' p g_k(k, p)}{p^2} \begin{cases} 
= 0, & \text{if } \delta'(c) = 0 \\
< 0, & \text{if } \delta'(c) > 0 
\end{cases}
\]

A diagrammatic representation of these functions is given in Figure 4.

Before deriving main results of this section, let us state the following lemma on the steady state:

**Lemma 4.** There is no the steady state in the region in which \( k \in [k_g(p, n), k_c(p, n)] \).

**Proof.** By the analysis above, \( \text{LHS}(k, p, n) \geq \delta_0 > n \), in which \( k \in [0, k_c(p, n)] \). On the other hand, \( \text{RHS}(k, p) \leq n \), in which \( k \in [k_g(p, n), k_c(p, n)] \). Therefore, \( \text{LHS}(k, p, n) > \text{RHS}(k, p) \) in which \( k \in [k_g(p, n), k_c(p, n)] \) necessarily. \( \square \)

First, we analyse the case \( \delta'(c) = 0 \). If we define \( \hat{p} \) as the relative price which satisfies

\[
\delta_0 - n = g_k(\hat{k}, \hat{p}) - (\mu + n)
= \hat{r}(\hat{p}) - (\mu + n),
\]

then we can derive the following proposition.

**Proposition 1.** If \( \delta'(c) = 0 \), then in the steady state

(i) there are a continuum of possible equilibria with diversified production and two possible equilibria specified to either good at only \( p = \hat{p} \),

(ii) there is unique equilibrium specialized to either good at \( \forall p \in (0, \hat{p}) \cup (\hat{p}, \infty) \).
Proof. (i) First, by \( \delta' (c) = 0 \), \( \text{LHS} (k, p, n) = \delta_0 \), \( \forall k \in [0, k_g (\hat{p}, n)) \), \( \forall p \in (0, \infty) \), and \( \forall n \in [0, \infty) \). Second, \( \text{RHS} (k, \hat{p}) \) is a continuous and monotonic decreasing function of \( k \), and has the following properties:

\[
\text{RHS} (k, \hat{p}) = g_k (k, \hat{p}) - \mu \begin{cases} 
\forall k \in [0, k_i (\hat{p})) \text{,} \\
> \delta_0, \\
= g_k (k_i (\hat{p}), \hat{p}) - \mu = \delta_0, \\
= \hat{r} (\hat{p}) - \mu = \delta_0, \\
= g_k (k_j (\hat{p}), \hat{p}) - \mu = \delta_0, \\
< \delta_0, \\
\forall k \in (k_i (\hat{p}), k_j (\hat{p})) \text{,} \\
\forall k \in (k_j (\hat{p}), k_g (\hat{p}, n)) \text{,}
\end{cases}
\]

where \( k_i (p(t)) < k_j (p(t)) \), \( i, j = 1, 2 \) and \( i \neq j \), because of the definition of \( \hat{p} \). Therefore, \( \forall k \in [k_i (\hat{p}), k_j (\hat{p})) \), \( \text{LHS} (k, \hat{p}, n) = \text{RHS} (k, \hat{p}) \), that is, there are a continuum of possible long-run equilibria with diversified production in \( \forall k \in (k_i (\hat{p}), k_j (\hat{p})) \) and two possible equilibria specified to either good in \( k = k_i (\hat{p}) \) or \( k_j (\hat{p}) \). Finally, from Eq. (4c'), function \( \hat{r} (p) \) is a monotonic function which satisfies \( \sup \hat{r} (p) = \infty \) and \( \inf \hat{r} (p) = 0 \). Therefore, \( \hat{p} \), defined as mentioned above, is uniquely determined. (ii) From the analysis of Section 2.1, when \( \delta' (c) = 0 \), \( \forall p \in (0, \hat{p}) \) (resp. \( \forall p \in (\hat{p}, \infty) \)), this economy is specialized in good 2 (resp. good 1) and \( g_{kk} (k, p) < 0 \) in the steady state. Therefore, \( k \) satisfies that \( \text{LHS} (k, p, n) = \text{RHS} (k, p) \) is unique \( \forall p \neq \hat{p} \), because \( \text{RHS} (k, p) \) is continuous and monotonic decreasing function and \( \partial \text{RHS} (k, p)/\partial k < 0 \) in the steady state.

This proposition is same as that of Stiglitz (1970) for small country case, Woodland (1982) and Chen (1992, Proposition 5) for two-country case, fundamentally.

Next, we can derive the proposition in the case of \( \delta' (c) > 0 \) as follows:

**Proposition 2.** If \( \delta' (c) > 0 \), \( \forall p \in (0, \infty) \), then there is unique equilibrium in the steady state.

**Proof.** By the analysis above, \( \lim_{k \downarrow 0} \text{LHS} (k, p, n) = \delta_0 < \lim_{k \uparrow 0} \text{RHS} (k, p) = \infty \) and \( \lim_{k \uparrow k_g (p, n)} \text{LHS} (k, p, n) > \lim_{k \uparrow k_g (p, n)} \text{RHS} (k, p) = n \). Moreover, because \( \text{LHS} (k, p, n) \) is a strictly monotonic increasing function and \( \text{RHS} (k, p) \) is a monotonic decreasing function of \( k \), there is unique steady state equilibrium in the region in which \( k \in (0, k_g (p, n)) \) necessarily.

Finally, we analyze the steady state effect of an increase in the relative price \( p \) and the population growth rate \( n \). Differentiating (16) using \( \delta (\bar{c}) = \delta_0 + \theta (\bar{c}) \) and \( \theta' (\bar{c}) = \delta' (\bar{c}) \), we can
obtain the system by the following expressions:

\[
\begin{bmatrix}
\delta' & -g_{kk} \\
p & -[g_k - (\mu + n)]
\end{bmatrix}
\begin{bmatrix}
de \\
dk
\end{bmatrix} =
\begin{bmatrix}
g_{kp} & 0 & -1 \\
[g_p - \bar{c}] & \bar{k} & 0
\end{bmatrix}
\begin{bmatrix}
dp \\
dn \\
d\delta_0
\end{bmatrix}.
\]

Therefore, the steady state effect of an increase in \( p \) is obtained as follows:

\[
\frac{\partial \bar{c}}{\partial p} = \left[ \frac{g_k - (\mu + n)}{\delta'[g_k - (\mu + n)] - pg_{kk}} \right] \frac{g_{kp} - \bar{c}}{\delta'[g_k - (\mu + n)] - pg_{kk}},
\]

\[
\frac{\partial \bar{k}}{\partial p} = \left[ \frac{g_k - (\mu + n)}{\delta'[g_k - (\mu + n)] - pg_{kk}} \right] \frac{pg_{kp} - \bar{c}}{\delta'[g_k - (\mu + n)] - pg_{kk}}.
\]

These equations allow for a straightforward exercise of basic microeconomic theory. The first term on the right hand side, \( [g_k - (\mu + n)]g_{kp}/(\delta'[g_k - (\mu + n)] - pg_{kk}) \), measures the substitution effect and is negative (resp. positive) when \( k_1(p) < k_2(p) \) (resp. \( k_1(p) > k_2(p) \)). Moreover, the second term, \( [g_p - \bar{c}]/(\delta'[g_k - (\mu + n)] - pg_{kk}) \), measures the income effect and is positive and negative (resp. negative and positive) when \( g_p > \bar{c} \) (resp. \( g_p < \bar{c} \)), respectively.

Especially, it is important to see the sign of \( \frac{\partial \bar{k}}{\partial p} \). Defining the elasticity \( \varepsilon_{\delta}(c) \equiv \delta'(c)c/\delta(c) \geq 0, \) we have

\[
\frac{\partial \bar{k}}{\partial p} = \left( \frac{g_k - (\mu + n)}{\delta'[g_k - (\mu + n)] - pg_{kk}} \right) \left( \frac{pg_{kp}}{g_k - (\mu + n)} - \frac{g_p - \bar{c}}{\bar{c}} \right) \varepsilon_{\delta},
\]

and therefore we can induce

\[
\left( \frac{g_p - \bar{c}}{\bar{c}} \right) \varepsilon_{\delta} \lesssim \frac{pg_{kp}}{g_k - (\mu + n)} \Leftrightarrow \frac{\partial \bar{k}}{\partial p} \gtrless 0.
\]

The signs of \( \frac{\partial \bar{k}}{\partial p} \) in the case of constant marginal impatience (\( \delta'(c) = 0 \)) and increasing marginal impatience (\( \delta'(c) > 0 \)) are summarized in Table 1 and 2, respectively.

[Table 1 about here.]
From these tables, in the cases of diversification or specialization to goods 1, if \( \delta'(c) > 0 \) and
\[
\left| \frac{g_p - \bar{c}}{\bar{c}} \right| \varepsilon_{\delta} > \left| \frac{pg_{kp}}{g_k - (\mu + n)} \right|,
\]
then \( \partial \bar{k}/\partial p \) may take opposite sign to that in the case of \( \delta'(c) = 0 \). Note that if consumption goods are traded internationally in large quantities or the elasticity of subjective discount rate with respect to consumption is large, the income effect is more likely to dominate the substitution effect. Since the assumption that the subjective discount rate is sufficiently inelastic with respect to consumption leads to preferences more comparable with time-separable preferences, which means \( \varepsilon_{\delta}(c) = 0 \), and more consistent with Assumption 5, we focus on the case which it is so inelastic that the income effect does not dominate the substitution effect, that is,
\[
\text{sign} \left( \frac{\partial \bar{k}}{\partial p} \right) = \text{sign} \left( \frac{pg_{kp}}{g_k - (\mu + n)} \right)
\]
in the case of diversification and specialization to good 1, in discussing the effect of the world relative price movements\(^{10}\). Therefore, we can summarize the effect of \( \bar{k} \) in increase in \( p \) as follows:
\[
\frac{\partial \bar{k}}{\partial p} = \begin{cases} 
< 0, & p \in (p_{\min}, p_{\max}) \\
> 0, & p \in (0, p_{\min}] \cup [p_{\max}, \infty) \\
> 0, & p \in (0, \infty)
\end{cases}, \text{if } k_1(p) < k_2(p), \forall p \in (0, \infty), \tag{20}
\]
where
\[
p_{\min} \equiv p_{\min}(n, \delta_0) = \min \{ P_1(n, \delta_0), P_2(n, \delta_0) \}, \\
p_{\max} \equiv p_{\max}(n, \delta_0) = \max \{ P_1(n, \delta_0), P_2(n, \delta_0) \},
\]
\(^{10}\)Note that we can make the absolute value of the income effect arbitrarily smaller than that of the substitution effect by choosing \( \varepsilon_{\delta}(c) \) sufficiently small. For example, when it is such that
\[
0 \leq \varepsilon_{\delta} \leq \frac{pg_{kp}/[g_k - (\mu + n)]}{\{pg_{kp}/[g_k - (\mu + n)]\} - 1}
\]
in the cases of diversification or specialization to goods 1, the substitution effect dominates the income effect necessarily.
and $P_i(n, \delta_0) \equiv \{ p : k(p, n, \delta_0) = k_i(p) \}, \ i = 1, 2.$

Finally, the effect of an increase in $n$ is obtained as follows:

$$\partial \bar{c} / \partial n = \bar{k} g_k - (\mu + n) - pg_{kk} \leq 0,$$

$$\partial \bar{k} / \partial n = \bar{k} \delta' [g_k - (\mu + n)] - pg_{kk} \geq 0,$$

$$\partial \delta_0 / \partial c = - \bar{k} \delta' = \delta' [g_k - (\mu + n)] - pg_{kk} < 0,$$

$$\partial \delta_0 / \partial \delta_0 = - p < 0.$$

3.2 Stability of Dynamic System

3.2.1 Stability in the case of the constant marginal impatience

Noting that $\delta(c) = \delta_0, \ \forall c \in [0, \infty)$ in the constant marginal impatient case, we have $v(c, \phi, n) = u(c) - (\delta_0 - n) \phi, \ v_c(c, \phi, n) = u'(c), \ and \ v_{cc}(c, \phi, n) = u''(c).$ Hence, Eq. (15a) reduces to

$$\hat{\sigma}(c) \equiv - u'(c) u''(c) c > 0.$$

Then, the dynamic system in the case of the constant marginal impatience is constructed by Eqs. (15a') and (15b). The following proposition characterizes a stability of this dynamic system when $\delta'(c) = 0.$

Proposition 3. Under the case of $\delta'(c) = 0,$

(i) if $p = \hat{p},$

(a) then a continuum of diversified steady states $(\bar{c}, \bar{k})$ which satisfies $\bar{k} \in (k_i(\hat{p}), k_j(\hat{p})),

k_i(p(t)) < k_j(p(t)), \ i, j = 1, 2, \ i \neq j,$ have no transitional dynamics,

(b) then the steady state $(\bar{c}, \bar{k}) = (\bar{c}, k_i(\hat{p}))$ or $(\bar{c}, k_j(\hat{p})),$ $i, j = 1, 2, \ i \neq j,$ which is specified to either goods, is locally saddle-path stable,

(ii) if $p \in (0, \hat{p}) \cup (\hat{p}, \infty),$ then the steady state $(\bar{c}, \bar{k}), \ which \ is \ specified \ to \ either \ goods, \ is \ locally \ saddle-path \ stable.$

Proof. Strictly speaking, the dynamic system (15a') and (15b) is classified into the following three types:
(i) $p = \hat{p}$ and

(a) $\forall k \in (k_i(\hat{p}), k_j(\hat{p}))$, $k_i(p(t)) < k_j(p(t))$, $i, j = 1, 2, i \neq j$,

$$\dot{c} = \hat{\sigma}(c) \{\dot{r}(\hat{p}) - \mu\} - \delta_0 \, c,$$

$$\dot{k} = [\hat{r}(\hat{p}) k + \dot{w}(\hat{p})] - (\mu + n) k - \hat{p} c,$$

(b) $\forall k \in [0, k_i(\hat{p})) \cup [k_j(\hat{p}), \infty)$, $k_i(p(t)) < k_j(p(t))$, $i, j = 1, 2, i \neq j$,

$$\dot{c} = \hat{\sigma}(c) \{\sigma_k(k, \hat{p}) - \mu\} - \delta_0 \, c,$$

$$\dot{k} = g(k, \hat{p}) - (\mu + n) k - \hat{p} c,$$

(ii) $\forall p \in (0, \hat{p}) \cup (\hat{p}, \infty)$,

$$\dot{c} = \hat{\sigma}(c) \{\sigma_k(k, p) - \mu\} - \delta_0 \, c,$$

$$\dot{k} = g(k, p) - (\mu + n) k - pc.$$

In the case of (i.a), we have that $c = \bar{c}$, $\forall t \in [0, \infty)$, because $\dot{r}(\hat{p}) - \mu = \delta_0$. Then, from (16b') we obtain

$$\bar{c} = \frac{[\dot{r}(\hat{p}) - (\mu + n)] \bar{k} + \dot{w}(\hat{p})}{\hat{p}}.$$

Hence, our dynamical system in this case reduces to a single linear differential equation with respect to $k$:

$$\dot{k} = [\dot{r}(\hat{p}) - (\mu + n)] k + \dot{w}(\hat{p}) - \hat{p} \bar{c} = [\dot{r}(\hat{p}) - (\mu + n)] (k - \bar{k}).$$

From $\dot{r}(\hat{p}) - (\mu + n) > 0$, $\forall k \in [0, \sigma_k(\hat{p}, n))$ and $\bar{k} > 0$, we must have $k = \bar{k}$, $\forall t \in [0, \infty)$ to satisfy the transversality condition. Therefore, a continuum of diversified steady states $(\bar{c}, \bar{k})$ have no transitional dynamics.

In the case (i.b), there is the steady state specialized to either goods from Proposition 3. If $\dot{c} = \dot{k} = 0$ with $k_1(\hat{p}) > k_2(\hat{p})$ (resp. $k_1(\hat{p}) < k_2(\hat{p})$), the steady state capital stock $\bar{k}$ satisfies

$$\delta_0 = g_k(\bar{k}, \hat{p}) - \mu = \begin{cases} \hat{p} f'_k(\bar{k}) - \mu, & \text{if } \bar{k} \in [k_1(\hat{p}), \infty) \quad (\text{resp. } \bar{k} \in [0, k_1(\hat{p}))), \\ f'_k(\bar{k}) - \mu, & \text{if } \bar{k} \in [0, k_2(\hat{p})) \quad (\text{resp. } \bar{k} \in [k_2(\hat{p}), \infty)). \end{cases}$$

$$\Rightarrow \bar{k} = \begin{cases} k_1(\hat{p}), & \text{if } \bar{k} \in [k_1(\hat{p}), \infty) \quad (\text{resp. } \bar{k} \in [0, k_1(\hat{p}))), \\ k_2(\hat{p}), & \text{if } \bar{k} \in [0, k_2(\hat{p})) \quad (\text{resp. } \bar{k} \in [k_2(\hat{p}), \infty)). \end{cases}$$
because the property $\hat{p} f_1' (k_1 (\hat{p})) = f_2' (k_2 (\hat{p})) = \hat{r} (\hat{p})$ and Assumption 1. Linearizing the system of differential equations around the steady state $(\bar{c}, \bar{k})$ yields

$$\begin{bmatrix} \dot{c} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} 0 & \bar{c} \sigma g_{kk} \\ -\hat{p} & g_k - (\mu + n) \end{bmatrix} \begin{bmatrix} c - \bar{c} \\ k - \bar{k} \end{bmatrix}$$

The characteristic equation associated with the matrix above is

$$I (\chi_i) \equiv \chi_i^2 - [g_k - (\mu + n)] \chi_i + \hat{p} \bar{c} \sigma g_{kk} = 0, \quad i = 1, 2$$

and the determinant and trace are given by

$$\begin{align*}
\chi_1 + \chi_2 &= g_k - (\mu + n) > 0, \\
\chi_1 \chi_2 &= \hat{p} \bar{c} \sigma g_{kk} = \hat{p} \bar{c} \sigma \frac{\partial \text{RHS}}{\partial k} < 0,
\end{align*}$$

because

$$g_{kk} (\bar{k}, \hat{p}) = \begin{cases} 
\hat{p} f_1'' (k_1 (\hat{p})) < 0, & \text{if } \bar{k} \in [k_1 (\hat{p}), \infty) \quad (\text{resp. } \bar{k} \in [0, k_1 (\hat{p})]), \\
f_2'' (k_2 (\hat{p})) < 0, & \text{if } \bar{k} \in [0, k_2 (\hat{p})] \quad (\text{resp. } \bar{k} \in [k_2 (\hat{p}), \infty)).
\end{cases}$$

Therefore, the steady state $(\bar{c}, \bar{k})$ is locally saddle-path stable.

In the case (ii), there is also the steady state specialized to either goods from Proposition 3. Thus, from the property in this case

$$g_{kk} (\bar{k}, p) = \begin{cases} 
p f_1'' (\bar{k}) < 0, & p \in (\hat{p}, \infty), \\
f_2'' (\bar{k}) < 0, & p \in (0, \hat{p})
\end{cases}$$

we have also the property of saddle path stability in the similar way as in the case (i.b).

Remark 2. Note that in Figure 4, if the LHS curve intersects the RHS curve from below, the steady state will be locally saddle-point stable. On the other hand, if the slopes of the LHS curve and the RHS curve are same, that is, $\partial \text{LHS}/\partial k - \partial \text{RHS}/\partial k = 0$, at the intersections, then the steady state will have no transitional dynamics.
We show the results above by using phase diagram in the \((k,c)\) plane in the case of \(p = \hat{p}^{11}\). 

\[ \dot{k} = 0 \]
is of inverted “U” shape with \(k\) axis intercepts 0 and \(k_c(p,n)\) and attains its maximum at \(k_g(p,n)\). On the other hand, if \(k_1(\hat{p}) < k_2(\hat{p})\) (resp. \(k_1(\hat{p}) > k_2(\hat{p})\)), then \(\dot{c} = 0\) locus is a region in which \(k \in [k_1(\hat{p}), k_2(\hat{p})]\) (resp. \(k \in [k_2(\hat{p}), k_1(\hat{p})]\)) (see Stiglitz (1970, pp.479-480)).

We illustrate the transitional dynamics in the case of \(k_1(\hat{p}) < k_2(\hat{p})\) in Figure 5, by considering different initial capital-labor ratio \(k_0\). First, if the economy begins with \(k_{01} \in (0, k_1(\hat{p})]\), it tend to grow to reach \(k_1(\hat{p})\), which corresponds to the specialization to goods 1 in the long-run. Second, if the country begins with \(k_{02} \in [k_1(\hat{p}), \infty)\), it converges to \(k_2(\hat{p})\), which corresponds to the specialization to goods 2. Finally, if the initial capital-labor ratio is \(k_{0d} \in (k_1(\hat{p}), k_2(\hat{p}))\), initial consumption \(c(0)\) jumps to the steady state level, that is, there is no transitional dynamics, and the conomy is incompletely specialized in the long-run.

\[\text{[Figure 5 about here.]}\]

3.2.2 Stability in the case of the increasing marginal impatience

Let us analyze the local stability property of the steady states in the case of \(\delta'(c) > 0\). Linearizing the system of differential equations (15) around the steady state yields

\[
\begin{bmatrix}
\dot{c} \\
\dot{k} \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
0 & \bar{c}\sigma g_{kk} & -\frac{\delta'v_{pk}}{v_{cc}} \\
-p & g_k - (\mu + n) & 0 \\
-v_c & 0 & g_k - (\mu + n)
\end{bmatrix}
\begin{bmatrix}
c - \bar{c} \\
k - \bar{k} \\
\phi - \bar{\phi}
\end{bmatrix}
\]

Evaluating the Jacobian of the system above at the steady state, we obtain the following characteristic equation:

\[
\mathcal{J}(\xi_i) \equiv [\xi_i - \{g_k - (\mu + n)\}] [\xi_i^2 - \{g_k - (\mu + n)\} \xi_i + \bar{c}\sigma \{pg_{kk} - \delta'[g_k - (\mu + n)]\}] = 0,
\]

(21)

\(^{11}\)See Stiglitz (1970, p.480, Figure A3) for the phase diagram in the case of the specialization to each goods, which corresponds to the case of \(p \in (0, \hat{p}) \cup (\hat{p}, \infty)\) in our analysis.
where $\xi_i$, $i = 1, \ldots, 3$ denote eigenvalues of the above equation, respectively. One of the characteristic roots, say $\xi_1$, is given by $g_k - (\mu + n) > 0$. Other characteristic roots, say $\xi_2 > \xi_3$, are the solution of the following quadratic equation:

$$\xi_i^2 - \{g_k - (\mu + n)\} \xi_i + p\bar{c}\sigma \left[ g_{kk} - \left( \frac{\delta'}{\bar{p}} \right) \{g_k - (\mu + n)\} \right] = 0. \quad (21')$$

Then, we can derive

$$\xi_2 + \xi_3 = g_k - (\mu + n) > 0,$$

$$\xi_2\xi_3 = p\bar{c}\sigma \left[ g_{kk} - \left( \frac{\delta'}{\bar{p}} \right) \{g_k - (\mu + n)\} \right] = p\bar{c}\sigma \left[ \frac{\partial \text{RHS}}{\partial k} - \frac{\partial \text{LHS}}{\partial k} \right] \geq 0.$$

Since $k$ is only state variable (with an initial condition), the steady state is saddle-point stable if $\xi_3 < 0 < \xi_2$, which happens if $\xi_2 \xi_3 < 0$, and it is unstable if $0 < \xi_3 < \xi_2$, which happens if $\xi_2 \xi_3 > 0$\(^{12}\).

The results above are summarized in the following theorem.

**Theorem 1.** If $\delta' (c) \neq 0$, $\forall c \in [0, \infty)$, and in a neighborhood of the steady state

$$\frac{\partial \text{LHS}}{\partial k} - \frac{\partial \text{RHS}}{\partial k} \begin{cases} < 0, & \text{then the non-trivial steady state will be unstable;} \\ > 0, & \text{then the non-trivial steady state will be saddle-point stable.} \end{cases}$$

**Remark 3.** Note that in Figure 4, if the LHS curve intersects the RHS curve from below (resp. above), the steady state will be locally saddle-point stable (resp. unstable).

Thus, Corollary 1 follows from Theorem 1:

**Corollary 1.** If $\delta' (c) > 0$, then the unique steady state is saddle-point stable, regardless of the relative price $p$, the initial capital stock $k_0$, the factor intensity in production sectors, and the specialization pattern in this economy.

As we have seen in Section 3.2.1, in the case of $\delta' (c) = 0$, we must have $k_0 \in (k_i (\hat{p}), k_j (\hat{p}))$, $k_i (p(t)) < k_j (p(t))$, $i, j = 1, 2$, $i \neq j$ and $k(t) = \bar{k}$, $\forall t \geq 0$, for the economy to satisfy the transversality conditions under incomplete specialization. On the other hand, in the case

\(^{12}\)Note that we can rule out the possibility of $\xi_3 < \xi_2 < 0$, that is, inderminaminacy, because $\xi_2 + \xi_3 > 0$.
of $\delta' (c) > 0$, given an initial capital stock $\forall k_0 \in (0, \infty)$, there exists unique stable path to convergent to the steady state under incomplete specialization.

We show the results above by using phase diagram in the $(k, c)$ plane. The dynamic system characterizing the optimal path is given by (15). These differential equations generates a three-dimensional phase diagram, which is intuitively difficult to analyze. We can nevertheless draw a two-dimensional phase diagram in the $(k, c)$ plane in the following way\(^{13}\).

\[ \dot{k} = 0 \] locus is just the same shape as it in Figure 5. On the other hand, $\dot{c} = 0$ locus is characterized by the following equation (see Appendix):

\[ g_k (k, p) - \mu = \delta (c) + \frac{\delta' (c) [g (k, p) - (\mu + n) k - pc]}{p}. \]

If $\delta' (c) > 0$, then $\dot{c} = 0$ is of inverted “C” shape and intersects the $\dot{k} = 0$ locus at $\bar{k}$, and at this point of intersection, the slope of it becomes infinity.

Figure 6 illustrates the optimal transitional path in the case of $\delta' (c) > 0$. Let us assume that there is initial capital stock $k_0$ in a region in which $k_0 \in (0, k_1 (p)]$. Then, the economy follows the stable path toward the steady state $(\bar{k}, \bar{c})$, while the pattern of specialization changes from the complete specialization in the production of the investment good to incomplete specialization over time. Such a change in the pattern of specialization differs from it in the case of $\delta' (c) = 0$.

[Figure 6 about here.]

\section*{4 International Trade}

Let us now analyze the international trade with the rest of world in this section. Following Stiglitz (1970) and Woodland (1982), we define the net imports of the investment good in the steady state as $m \equiv m (p, n, \delta_0)$, which satisfies

\begin{equation}
\begin{aligned}
m (p, n, \delta_0) &= (\mu + n) k (p, n, \delta_0) - [g (k (p, n, \delta_0), p) - pg_p (k (p, n, \delta_0), p)] \\
&= p [g_p (k (p, n, \delta_0), p) - c (p, n, \delta_0)].
\end{aligned}
\end{equation}

First, the effects of $m$ in increase in the price of the investment good in terms of the

\(^{13}\)We assume the case $p \in (p_{\text{min}}, p_{\text{max}})$, that is, the case of the steady state with diversified production. We can also illustrate the case $p \in (0, p_{\text{min}}] \cup [p_{\text{max}}, \infty)$ in the same way.
consumption good $1/p$ is obtained as follows:

$$\frac{\partial m}{\partial (1/p)} = -p^2 \frac{\partial m}{\partial p} = -p^2 \left[ pg_{pp} + \{ g_k - (\mu + n) \} \left\{ \frac{pg_{kp}}{g_k - (\mu + n)} - 1 \right\} \frac{\partial \tilde{k}}{\partial p} \right].$$

(23)

Furthermore, we define the autarkic price as

$$p_a \equiv p_a(n, \delta_0) = \{ p : (\mu + n) k(p, n, \delta_0) = g(k(p, n, \delta_0), n, \delta_0) - pg_p(k(p, n, \delta_0), n, \delta_0) \}.$$

Note that in the neighborhood of $m = 0$,

$$\frac{\partial m}{\partial (1/p)} \bigg|_{m=0} = -p^3 \left[ g_{pp} + g_{kp} \left\{ \frac{p_a g_{kp}}{g_k - (\mu + n)} - 1 \right\} \left\{ \frac{g_k - (\mu + n)}{\delta'(g_k - (\mu + n)) - g_{kk}} \right\} \right].$$

(23')

We can derive the following lemma on $p_a$.

**Lemma 5.** There is the autarkic price $p_a \in (p_{\text{min}}, p_{\text{max}})$ uniquely.

**Proof.** **Existence:** If $p \in (0, p_{\text{min}}]$, the economy specializes in the investment good, which means $m < 0$. Moreover, if $p \in [p_{\text{max}}, \infty)$, the economy specializes in the consumption good, which means $m > 0$. Finally, if $p \in (p_{\text{min}}, p_{\text{max}})$, then $\lim_{p \to p_{\text{min}}} m < 0$, $\lim_{p \to p_{\text{max}}} m > 0$. Because function $m$ is continuous in $p$, there exists at least one of $p_a$ such that $p_a \in (p_{\text{min}}, p_{\text{max}})$. **Uniqueness:** It is obvious $\frac{\partial m}{\partial (1/p)} \big|_{m=0} < 0$ because $p_a \in (p_{\text{min}}, p_{\text{max}})$. If there is not unique $p$ which satisfy $m = 0$, at least one of these necessarily satisfy $\frac{\partial m}{\partial (1/p)} \big|_{m=0} \geq 0$ because function $m$ is continuous. This is contradictory to Eq.(23'). Therefore, $p_a$ is unique.

We obtain the following theorem.

**Theorem 2.** Under Assumptions 1-6,

$$m = (\mu + n) \bar{k} - \left[ g(\bar{k}, p) - pg_p(\bar{k}, p) \right] \begin{cases} > 0, & p \in (p_a, \infty), \\ = 0, & p = p_a, \\ < 0, & p \in (0, p_a). \end{cases}$$

Figure 7 illustrates the relations between $m$ and $1/p$ in the case $\delta'(c) > 0$ and satisfies Eq.(20)\textsuperscript{14}.

\textsuperscript{14}See Stiglitz (1970) for the case $\delta'(c) = 0$. 26
Second, the effects of $m$ in increase in the population growth rate $n$ is obtained as follows:

$$\frac{\partial m}{\partial n} = \bar{k} + \left[ pg_{kp} (k, p) - \{g_k (k, p) - (\mu + n)\} \right] \frac{\partial k}{\partial n} > 0.$$ 

Therefore, the increase in the population growth rate shifts $mm$ locus rightward in Figure 7.

Finally, the effects of $m$ in increase $\delta_0$ is obtained as follows:

$$\frac{\partial m}{\partial \delta_0} = p \left[ pg_{kp} (k, p) - \{g_k (k, p) - (\mu + n)\} \right] \begin{cases} > 0, & k_1 (p) < k_2 (p) \\ < 0, & k_1 (p) > k_2 (p) \end{cases}, \forall p \in (0, \infty).$$

Therefore, if good 2 is more capital-intensive than good 1, the increase in $\delta_0$ shifts $mm$ locus rightward in Figure 7, as same as the case of the increase in $n$.

5 Decreasing Marginal Impatience

In this section, we analyze the properties in the long run when households have decreasing marginal impatience, i.e., the discount function is negatively related to the consumption of each time. Das (2003) demonstrated that unique saddle-point steady state could be consistent with decreasing marginal impatience. We assume the assumption on the subjective discount function $\delta (c(t))$ as follows:

**Assumption 7** (the subjective discount rate function with decreasing marginal impatience). The subjective discount rate function $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$ is real valued, bounded above, $\inf \delta (c(t)) \equiv \underline{\delta} > n$, and twice continuously differentiable with $\delta' (c(t)) < 0$ and $\delta'' (c(t)) > 0$, $\forall c(t) \in [0, \infty)$.

**Remark 4.** Note that Assumption 7 follows Das (2003), except that we make an additional assumption that $\delta (c(t)) > n$. Moreover, by Assumption 3 and 7, $\rho (c(T), U (T \cdot c), n) > 0, \forall c(t) \in [0, \infty)$. Unlike the case of the “increasing marginal impatience”, however, note also that $\partial \rho (c(T), U (T \cdot c), n)/\partial U (T \cdot c) < 0, \forall c(t) \in [0, \infty)$, i.e., an increase in the lifetime utility makes people more patient, and in “ceteris paribus” induces them to consume less at present and save more. This property does not necessarily seem to be plausible intuitively.
In order to satisfy Assumption 7, the function $\theta(c(t))$ in Eq. (6) requires that $\theta(0) = 0$, $\delta_0 + \inf \theta(c(c(t))) > n$, $\theta'(c(t)) = \delta'(c(t)) < 0$ and $\theta''(c(t)) = \delta''(c(t)) > 0$, $\forall c(t) \in [0, \infty)$, in this section (See Figure 2).

Chang (2004), which is one-sector optimal growth model in a closed economy, assumed the following inequality to ensure the uniqueness of the steady state:

$$
\delta'(0) \geq \left( \frac{1}{\delta_0} \right) f''(k), \quad \forall k \in \left[ f'^{-1}(\delta_0 + \mu), f''^{-1}(\delta + \mu) \right].
$$

We impose the following assumption as a corresponding condition to the inequality above.

**Assumption 8** (Bounded slope of the function $\delta(c(t)))$.

$$
\delta'(0) \geq \left( \frac{p}{\delta_0} \right) \max \left\{ pf''_1(k) , f''_2(k) \right\}, \quad \forall k \in \left[ k_{\min} , k_{\max} \right],
$$

where $k_{\min} = \min \left\{ pf'_{1}^{-1}\left( (\delta_0 + \mu)/p \right) , f'_{2}^{-1}(\delta_0 + \mu) \right\}$ and $k_{\max} = \max \left\{ pf'_{1}^{-1}\left( (\delta + \mu)/p \right) , f'_{2}^{-1}(\delta + \mu) \right\}$, $\forall p \in (0, \infty)$.

Except for using Assumption 7 and 8 instead of Assumption 4, the other assumptions and optimal conditions in this section are same as those in Section 2 and 3. Note that in the case of decreasing marginal impatience the properties of function LHS$(k, p, n)$ (Eq. (18b')) are given by

$$
\lim_{k \downarrow 0} \text{LHS}(k, p, n) = \text{LHS}(k_c(p, n), p, n) = \delta_0,
$$

$$
\frac{\partial \text{LHS}(k, p, n)}{\partial k} = \frac{\delta'}{p} \left[ g_k(k, p) - (\mu + n) \right] \begin{array}{l} < 0, \quad \text{if } k \in [0, k_g(p, n)) , \\
= 0, \quad \text{if } k = k_g(p, n), \\
> 0, \quad \text{if } k \in (k_g(p, n), k_c(p, n)) , \end{array}
$$

$$
\frac{\partial^2 \text{LHS}(k, p, n)}{\partial k^2} = \frac{\delta'' \left[ g_k(k, p) - (\mu + n) \right]^2 + \delta' g_{kk}(k, p)}{p^2} > 0.
$$

Then, we can derive the following proposition on the property of steady state:

**Proposition 4.** If $\delta'(c) < 0$, then

(i) the economy has at least one non-trivial steady state,
(ii) \( \forall p \in (0, p_{\text{min}}) \cup (p_{\text{max}}, \infty) \), there is unique long-run equilibrium specialized to either good and \( \frac{\partial \text{LHS}}{\partial k} > \frac{\partial \text{RHS}}{\partial k} = 0 \) at this point,

(iii) \( \forall p \in (p_{\text{min}}, p_{\text{max}}) \), there are multiple long-run equilibria, that is, \( \tilde{k}_i(p) \in (0, k_i(p)) \) which \( \frac{\partial \text{LHS}}{\partial k} > \frac{\partial \text{RHS}}{\partial k}, \tilde{k}_d(p) \in (k_i(p), k_j(p)) \) which \( \frac{\partial \text{LHS}}{\partial k} < \frac{\partial \text{RHS}}{\partial k} = 0 \), and \( \tilde{k}_j(p) \in (k_j(p), \infty) \) which \( \frac{\partial \text{LHS}}{\partial k} > \frac{\partial \text{RHS}}{\partial k}, k_i(p(t)) < k_j(p(t)), i, j = 1, 2, i \neq j \), respectively.

Using Theorem 1 and Proposition 4, we obtain the following corollary on the stability of the steady state.

**Corollary 2.** If \( \delta'(c) < 0 \), then

(i) the unique equilibrium specialized to either good has the property of saddle point stable,

(ii) the incomplete specialization equilibrium has the property of unstable.

A diagrammatic representation of these propositions is given in Figure 8. In this figure, there is an incomplete specialization equilibrium, while the other are equilibria specialized to either good at \( \tilde{p} \in (p_{\text{min}}, p_{\text{max}}) \). By the analysis above, if the initial capital stock \( k_0 \neq \tilde{k}_d(\tilde{p}) \), the economy converges to the steady state specialized to either good, say \( E''_1 \) or \( E''_2 \), over time. As is well known, the phenomenon in this case is called “poverty trap” \(^{16}\). Poor countries in initial endowments, \( k_0 \), converge to a low steady state while rich countries converge to a high one, even though all countries share identical technologies and preferences in this setting.

[Figure 8 about here.]

6 Conclusion

In this paper, we presents a dynamic small country model of international trade with variable marginal impatience.

In the case of the increasing marginal impatience, the steady state with incomplete specialization exhibits uniqueness and saddle-point stability. On the other hand, in the case of the

\(^{16}\)Deardorff (2001) analyses a multisector neoclassical growth model in the small open economy and yeilds multiple steady states and thus poverty trap. The result in this section is essentially same as that of Deardorff (2001).
decreasing marginal impatience, contrary to the closed one-sector economy, the steady state
with incomplete specialization does not exhibit uniqueness and saddle-point stability. These
properties is different from the result in the case of the constant marginal impatience. Further-
more, we derive the properties of the trade pattern in the steady state. Especially, the property
in the increasing marginal impatience is well-behaved.

We conclude the paper by suggesting directions for further research. First, we need to
analyze the relationship between the trade equilibrium and the variable, especially decreasing,
marginal impatience. Second, it would be interesting to discuss about the gains from trade in
the current framework. Third, we need to derive the effect of the trade policy. Finally, it is
important to extend to the endogenous growth theory. These are problems that remain to be
examined.

Appendix

We characterize the property of the \( \dot{c} = 0 \) locus in the \((k, c)\) plane. Along an optimal trajectory,
all of the first-order conditions have to be satisfied. Then, in the present value Hamiltonian
(12), we can derive

\[
\lim_{t \to \infty} H = \lim_{t \to \infty} u(c) \exp \Delta + \lim_{t \to \infty} \Lambda [g(k, p) - (\mu + n) k - pc] - \lim_{t \to \infty} \Phi [\delta(c) - n] \\
= \lim_{t \to \infty} [u(c) - \phi(\delta(c) - n)] \exp \Delta + \lim_{t \to \infty} \Lambda [g(k, p) - (\mu + n) k - pc] \\
= \lim_{t \to \infty} v(c, \phi, n) \exp \Delta + \lim_{t \to \infty} \Lambda [g(k, p) - (\mu + n) k - pc] \\
= v(\bar{c}, \bar{\phi}, n) \lim_{t \to \infty} \exp \Delta + \lim_{t \to \infty} \Lambda [g(k, p) - (\mu + n) k - pc] \\
= 0 \cdot \lim_{t \to \infty} \exp \Delta + 0 \cdot \lim_{t \to \infty} \Lambda = 0.
\]

Using \( \partial H / \partial c = 0, \partial H / \partial k = -\dot{\Lambda}, \partial H / \partial \Delta = -\dot{\Phi}, \partial H / \partial \Lambda = \dot{k}, \) and \( \partial H / \partial \Phi = \dot{\Delta}, \)

\[
\mathcal{H}(t) = -\int_t^\infty d\mathcal{H} = -\int_t^\infty \left[ \frac{\partial \mathcal{H}}{\partial c} \dot{c} + \frac{\partial \mathcal{H}}{\partial k} \dot{k} + \frac{\partial \mathcal{H}}{\partial \Delta} \dot{\Delta} + \frac{\partial \mathcal{H}}{\partial \Lambda} \dot{\Lambda} + \frac{\partial \mathcal{H}}{\partial \Phi} \dot{\Phi} \right] dt = 0, \quad \forall t \geq 0. \quad (A2)
\]

\(^{17}\)See Micheal (1982) and Barro and Sala-i-Martin (1995) for further details.
Therefore, we have

\[
\phi = \frac{u(c) + \lambda [g(k, p) - (\mu + n) k - pc]}{\delta(c) - n} = \frac{u(c) + u'(c) [g(k, p) - (\mu + n) k - pc]}{p [\delta(c) - n] + \delta''(c) [g(k, p) - (\mu + n) k - pc]}.
\]

(A3)

Using this equation and Eq.(15a), we can derive the \( \dot{c} = 0 \) locus,

\[
g_k(k, p) - \mu = \delta'(c) \frac{[g(k, p) - (\mu + n) k - pc]}{p}.
\]

(A4)

Differentiating both side of (A4), we get the slope of the \( \dot{c} = 0 \) locus,

\[
\frac{dc}{dk} = \frac{pg_{kk}(k, p) - \delta''(c) [g_k(k, p) - (\mu + n)]}{\delta''(c) [g(k, p) - (\mu + n) k - pc]}.
\]

(A4)

Therefore, we obtain \( dc/dk \geq 0 \) as \( \dot{k} = g(k, p) - (\mu + n) k - pc \geq 0 \).
References


Figures

Figure 1: GDP function \((k_1(p(t)) < k_2(p(t)))\)
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Table 1: the sign of $\partial \bar{k}/\partial p$ in the case of $\delta'(c) = 0$ and $k_1(p) < k_2(p)$ ($k_1(p) > k_2(p)$)

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Table 2: the sign of $\partial \bar{k}/\partial p$ in the case of $\delta'(c) > 0$ and $k_1(p) < k_2(p)$ ($k_1(p) > k_2(p)$)