Abstract

How does the sectoral relationship specificity affect the market thickness? How does the market thickness respond to opening international trade? To address these questions, we develop a dynamic industry model in which the relationship specificity endogenously pins down the market thickness in a general-equilibrium setting. Downstream firms match with upstream firms to obtain a customized component, and matched pairs can produce higher-quality products relative to unmatched counterparts in industries that rely on relationship-specific investments to raise the quality of a final good. We find that in a closed economy, an industry with higher relationship specificity has a thinner market. In an open economy, opening trade (or the reduction of trade costs) in final goods leads to a thinner market, and this effect is greater the higher is the relationship specificity.

Very preliminary and incomplete

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1 Introduction

The volume of international trade has been steadily rising over the past fifty years. It is often argued that offshoring in components especially contributes to this growth of trade volume, although trade in final goods still constitutes a large part of world trade. For instance, using input-output tables for developed and developing countries, Hummels et al. (2001) estimate that such offshoring accounts for around 30 percent of countries’ export growth between 1970 and 1990. Similarly, Hanson et al. (2005) find that international outsourcing for components has grown rapidly within U.S. multinationals, while final-good trade is nevertheless made up of a majority of the U.S. manufacturing exports. These pieces of empirical evidence suggest that offshoring in components and international trade in final goods have complementary effects on recent trade flows and their close interaction is a key to understanding the interdependence of countries via the fragmentation of production.

How does the sectoral relationship specificity affect the market thickness? How does the market thickness respond to opening international trade? To address these questions, we develop a dynamic industry model in which relationship specificity endogenously determines the market thickness in a general-equilibrium setting. In our model of vertical specialization, downstream firms randomly match with upstream firms to obtain a customized component. Matched upstream firms make relationship-specific investments for the needs of matched downstream firms, and these matched pairs can produce a high-quality final good. On the contrary, unmatched upstream firms manufacture a non-customized, generic component and sell it in the market to unmatched downstream firms, and these unmatched pairs inevitably produce a low-quality final good. The importance of such investments (i.e., relationship specificity) varies across industries and this difference of product quality is greater in the industries that rely more on relationship-specific investments. As a result, the market thickness that governs matching frictions between upstream and downstream firms is significantly affected by relationship specificity, which in turn affects the efficiency of final-good production. In this environment, once the economy is open to trade with a foreign country, offshoring in the components becomes an option for final-good producers and offshoring in components can have a significant impact on the market thickness and the quality distribution of the final goods.

Our main results are summarized as follows. In a closed economy, we find that an industry with higher relationship specificity has a “thinner” market in which the volume of components traded in the market is small relative to the volume of components traded directly inside of matched pairs of upstream and downstream firms. Profits of matched pairs are also high relative to those of unmatched pairs in such an industry. In an open economy, we consider two types of trade (final-good trade and component trade or offshoring) and examine their interaction as to the effect on the market thickness. Regarding final-good trade where only matched downstream firms, which produce a high-quality final good, are able to export, we show that the reduction of trade costs leads to a thinner market in any industry. Furthermore, this effect on the market thickness is greater, the higher is the relationship specificity.

Our paper is closely related to Nunn (2007). Devising a measure of input customization that
is a proxy for relationship specificity, he shows empirical evidence suggesting that countries with
better contract enforcement export relatively more in industries for which relationship-specific
investments are more important. While the focus on the relationship specificity is similar, his
analysis is confined to final-good trade only and hence he does not investigate how offshoring in
components interacts with international trade in final goods, which is a key research question in
the current paper. In somewhat different contexts, McLaren (2000) and Grossman and Helpman
(2002) also explore the impact of offshoring or outsourcing in component trade on the pattern
of specialization and trade. In these studies, however, the market thickness is exogenously given
and hence matching frictions between upstream and downstream firms are not affected by the
reduction of trade costs. By contrast, the market thickness is endogenous with the consequence
of globalization in our setup, thereby generating richer predictions.

2 Model

2.1 Demand

Consider an economy consisting of a number of sectors \( j \in \{1, 2, ..., J\} \). Each sector produces a
differentiated good in a monopolistically competitive market. The preferences of a representative
consumer are given by

\[
U = \sum_{j=1}^{J} \beta_j \log u_j, \quad \sum_{j} \beta_j = 1, \quad \beta_j > 0,
\]

where \( \beta_j \) is the share of the consumer’s income that is devoted into a differentiated good in
sector \( j \). Each lower-tier subutility for sector \( j \) is a Dixit-Stiglitz C.E.S. form:

\[
\phi_j \equiv \frac{\int_{\omega \in \Omega_j} \alpha_j(\omega)^{\sigma_j} x_j(\omega)^{\frac{\sigma_j - 1}{\sigma_j}} d\omega}{\sigma_j^{\frac{\sigma_j - 1}{\sigma_j}}}, \quad \sigma_j > 1.
\]

\( x_j(\omega) \) is consumption of variety \( \omega \) and \( \alpha_j(\omega) \) is quality of variety \( \omega \) such that the greater \( \alpha_j(\omega) \),
the higher the quality of \( \omega \) and the greater demand for that variety. This quality is sector-specific
in that its value differs across sectors but is the same across varieties within each sector.

The economy is endowed with \( L \) units of labor, which is a unique factor of production and is
completely mobile across sectors. We choose labor as a numeraire in the model, so that the wage
rate is equal to one \( (w = 1) \). Since free entry drives expected pure profits to zero, a country’s
aggregate labor income equals labor endowments \( L \) in general equilibrium. As is well-known,
this preference gives demand for variety \( \omega \) in sector \( j \):

\[
x_j(\omega) = E_j P_j^{\sigma_j - 1} \alpha_j(\omega) p_j(\omega)^{-\sigma_j},
\]

where \( E_j \equiv \beta_j L \) is aggregate expenditure in sector \( j \) and

\[
P_j = \left[ \int_{\omega \in \Omega_j} \alpha_j(\omega)p_j(\omega)^{1-\sigma_j} d\omega \right]^{\frac{1}{1-\sigma_j}}
\]
is the dual price index associated with the lower-tier subutility. In what follows, we focus on a particular sector and drop subscript $j$ for notational simplicity.

2.2 Production

A representative differentiated-good sector has upstream and downstream stages of production. In each production stage, firms are either unmatched or matched with firms from another stage of production. Regardless of their matching status, firms have the same production technology: one unit of component production requires $c$ units of labor in the upstream stage, whereas one unit of final-good production requires one unit of component in the downstream stage.

For unmatched firms, we assume that upstream firms sell their component in a perfectly competitive market and downstream firms buy this component from the market. The price of a component provided by unmatched upstream firms, $q$, thus equals the marginal cost $c$, and the profits for unmatched downstream firms are $\pi = (p - q)x = (p - c)x$. In addition, due to lack of partnership, upstream firms inevitably manufacture a standardized, generic component that is usable for any downstream firms with low quality, which leads unmatched pairs to capture small demand for their final good. In particular, the quality for these pairs satisfies $\alpha = 1/\gamma$, where $\gamma \in (1, \infty)$ denotes the degree of relationship specificity that varies across sectors. The greater $\gamma$, the lower the quality of a final good produced by unmatched pairs and hence the lower demand these pairs can capture in the final-good market.

For matched firms, by contrast, a component is transacted within pairs (not through the market). Once matched with partners, upstream firms bargain over profit sharing and, if they agree on it, these firms make a relationship-specific investment to customize a component for the needs of downstream firms. As a result, a component transacted within matched pairs becomes special and distinct for a particular matched pair, and not only do downstream firms but also upstream firms earn non-zero profits. These profits are respectively denoted by $\tilde{\pi}^D = (\tilde{p} - \tilde{q})\tilde{x}$ and $\tilde{\pi}^U = (\tilde{q} - c)\tilde{x}$, where tildes (˜) will be used to distinguish the variables of matched pairs from those of unmatched pairs. Because a component are now customized by the investment, matched pairs are also able to produce a high-quality final good relative to unmatched counterparts. We formalize this aspect by setting the product quality for matched pairs to $\tilde{\alpha} = 1$, so that the difference of product quality between matched and unmatched pairs is greater in sectors that rely on higher relationship specificity $\gamma$. While matched pairs negotiate on profit sharing, bargaining is ex-ante efficient without any uncertainty and thus they can choose the price to maximize their joint profits, $\tilde{\pi} = \tilde{\pi}^D + \tilde{\pi}^U$.

Both matched and unmatched pairs choose their price to maximize their profits, $\pi$ and $\tilde{\pi}$. Given the C.E.S. preferences, these pairs charge a price with a constant markup $\sigma/(\sigma - 1)$ over the marginal cost of component production $c$. Since $c$ is the same for matched and unmatched pairs, the pricing rules are also the same, $\tilde{p} = p = \sigma c/(\sigma - 1)$. Hence, the price index $P$ can be rewritten as

$$P = \left[ n\tilde{p}^{1-\sigma} + (N - n)\frac{\tilde{p}^{1-\sigma}}{\gamma} \right]^{\frac{1}{1-\sigma}} = \frac{\sigma c}{\sigma - 1} \left( n + \frac{N - n}{\gamma} \right)^{\frac{1}{1-\sigma}},$$
where \( n \) is the number of matched firms and \( N - n \) is the number of unmatched firms in the downstream stage, both of which are endogenously determined in the model (\( N \) is the total number of downstream firms). Moreover, from the price index and isoelastic demand derived above, we have the equilibrium output and profit levels for matched and unmatched pairs:

\[
x = \frac{\sigma - 1}{\sigma c} \frac{E}{\gamma (n + N - n)}, \quad \pi = \frac{(p - c)x}{\gamma \sigma (n + N - n)}; \\
\tilde{x} = \frac{\sigma - 1}{\sigma c} \frac{E}{n + N - n}, \quad \tilde{\pi} = \frac{(\tilde{p} - c)\tilde{x}}{\sigma (n + N - n)}.
\]

(1)

Hence the ratios of output and profit levels depend only on relationship specificity (\( \tilde{x} / x = \tilde{\pi} / \pi = \gamma \)): matched pairs produce outputs and earn profits relatively more than unmatched counterparts in sectors with higher relationship specificity.

### 2.3 Search technology

There are \( M \) upstream firms and \( N \) downstream firms, where the number of firms in each stage of production is endogenously pinned down by free entry. Upon paying fixed entry costs denoted by \( F_U \) and \( F_D \) (measured in units of labor), upstream and downstream firms respectively enter the component and final-good markets without knowing their matching status. These firms find themselves being either matched or unmatched after the entry cost becomes sunk. In the latter case, upstream firms seek potential downstream partners and downstream firms seek potential upstream partners, and they randomly match with each other. For tractability, we assume that one-to-one matching takes place in search: one upstream firm can successfully matches with only one downstream firm and vice versa. Not all firms are successful in their searches, however, and only a fraction of \( n \leq \min\{M, N\} \) firms are able to find its pairs among \( M \) upstream firms and \( N \) downstream firms. The number of matched pairs, \( n \), is also endogenously determined in the model by the search technology that allows \( n = \nu(M - n, N - n) \) pairs to be formed after search, where its arguments, \( M - n \) and \( N - n \), respectively denote the number of unmatched upstream firms and that of unmatched downstream firms that exist in the economy before search. Following the literature (e.g., Grossman and Helpman, 2002), we assume that this technology satisfies constant-returns-to-scale in matching, i.e., \( \nu(ka, kb) = k\nu(a, b) \) for any positive number of \( a \), \( b \) and \( k \). Put differently, a doubling in the number of firms on each side of production leads to a doubling in the number of matched pairs. We also assume that \( \nu(\cdot, \cdot) \) is increasing and concave in both of its arguments. These properties of search technology also jointly imply complementarity or supermodularity in matching. As originally studied by Shimer and Smith (2000), supermodular functions are routinely used in the matching and searching environments.\(^2\)

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\(^1\)Note that these profits of matched and unmatched pairs include neither the relationship-specific investment cost nor fixed entry cost. As will be clear later, \( \tilde{x} \) and \( \tilde{\pi} \) are the per-period profit levels before accounting for these fixed costs in the current dynamic model.

\(^2\)By supermodularity, we mean that \( \partial^2\nu(a, b)/\partial a \partial b > 0 \) for a strictly positive and twice differentiable \( \nu(a, b) \). Shimer and Smith (2000) use a slightly different definition of supermodularity, i.e., \( \nu(a, b) + \nu(a' + b') > \nu(a', b) + \nu(a, b') \) for any \( a > a' \) and \( b > b' \).
Using the constant-returns-to-scale property of the search technology, the probabilities of a match among upstream and downstream firms are respectively defined as

\[ \mu_D = \nu(M - n, N - n) = \nu \left( \frac{M - n}{N - n}, 1 \right), \]
\[ \mu_U = \nu(M - n, N - n) = \nu \left( 1, \frac{N - n}{M - n} \right) = \frac{N - n}{M - n} \mu_D. \] (2)

Every upstream firm has the same probability of being matched with a downstream firm and \( \mu_U \) is the same for all upstream firms; similarly, every downstream firm has the same probability of being matched with a upstream firm and \( \mu_D \) is the same for all downstream firms. However, these probabilities of a match are generally not the same (\( \mu_U \neq \mu_D \)). Let \( z \equiv (M - n)/(N - n) \) denote the ratio of the number of unmatched upstream firms relative to that of unmatched downstream firms. This ratio is an endogenous variable as the number of matched and unmatched firms is endogenously determined in the model. The above probabilities of realizing a match are then expressed in terms of only \( z \):

\[ \mu_D = \nu \left( \frac{M - n}{N - n}, 1 \right) \equiv s(z), \quad \mu_U = \frac{s(z)}{z}. \] (2')

From the properties of the search technology \( \nu(\cdot, \cdot) \), it follows that \( s(z) \) increases with \( z \), while \( s(z)/z \) decreases with \( z \). Further, \( s(z) \) is concave in \( z \). The ratio of unmatched pairs \( z \) and the normalized probability \( s(z) \) will play a key role in the equilibrium analysis.

To explicitly explore a search process in the upstream and downstream production stages, we study a simple dynamics in which matching and dissolving of firms are stochastically defined over time. Figure 1 illustrates this search process that occurs in the economy. In every period, there exist \( n \) matched firms who produce high-quality products. The remaining firms – \( M - n \) firms in the upstream stage and \( N - n \) firms in the downstream stage – are unmatched who produce low-quality products. Among matched pairs, some of them face a bad shock of dissolving the relationship; at the same time, among unmatched pairs, some of them are successful in matching. For the dissolving side, a fraction \( \lambda \) of matched pairs \( n \) break down exogenously, yielding \( \lambda n \) firms that fail to continue their partnership in every period. For the matching side, a fraction \( \mu_U \) of unmatched upstream firms \( M - n \) find their partner while a fraction \( \mu_D \) of unmatched downstream firms \( N - n \) find their partner, yielding \( \mu_U(M - n) \) upstream firms and \( \mu_D(N - n) \) downstream firms that start to form their partnership in every period.
We only consider a stationary equilibrium in which all endogenous variables (including the number of matched and unmatched firms) remain constant over time. In the steady state, the number of pairs that break relationships must be equal to the number of pairs that are newly formed. Consequently, for the pool of each production stage, we have

\[
\begin{cases}
\lambda_n = \mu^U(M - n), \\
\lambda_n = \mu^D(N - n).
\end{cases}
\]

In this steady-state relationship, while the rate of dissolving the relationship \( \lambda \) is exogenous, the total numbers of firms \( M \) and \( N \) are endogenously determined by free entry in the model. The number of matched pairs \( n \) and the probabilities of a match \( \mu^U \) and \( \mu^D \) are also endogenous. Further, solving the above steady-state relationship for \( n \) in each stage yields

\[
n = \left( \frac{\mu^U}{\lambda + \mu^U} \right) M = \left( \frac{\mu^D}{\lambda + \mu^D} \right) N.
\]

Equation (3) describes how a fraction of matched pairs is tied to the matching and breakdown probabilities in the dynamics.

To summarize, the sequence of events in the economy is as follows. First, firms enter either the upstream or downstream markets and pay the relevant fixed entry costs. Second, firms search for a potential partner from another stage of production, whereby matching occurs randomly between them. Finally, bargaining over the joint surplus takes place within matched pairs and monopolistic competition occurs across matched and unmatched pairs. In this setting, we seek a general equilibrium in which all product markets and aggregate labor market clear.

3 Closed-economy equilibrium

This section first describes a closed-economy version of the model. The next section then allows this economy to engage in final-good trade and examines the impact of trade.

3.1 Equilibrium conditions

To derive the closed-economy equilibrium in the current dynamic model, we define the Bellman equations in both stages of production. Let \( \tilde{V}^D \) and \( V^D \) respectively denote the value functions of matched and unmatched pairs in the downstream stage; similarly, let \( \tilde{V}^U \) and \( V^U \) denote those of matched and unmatched pairs in the upstream stage. Then, these value functions must satisfy the following Bellman equations:

\[
\begin{align*}
 r\tilde{V}^D &= \tilde{\pi}^D + \lambda \left( V^D - \tilde{V}^D \right) + \dot{\tilde{V}}^D, \\
 rV^D &= \pi + \mu^D \left( \tilde{V}^D - V^D \right) + \dot{V}^D, \\
 r\tilde{V}^U &= \tilde{\pi}^U + \lambda \left( V^U - \tilde{V}^U \right) + \dot{\tilde{V}}^U, \\
 rV^U &= \mu^U \left( \tilde{V}^U - K - V^U \right) + \dot{V}^U,
\end{align*}
\]
where \( r \) is a common discount factor and \( K \) (with \( K'(\gamma) \geq 0 \)) is relationship-specific investments. \( \tilde{\pi}^D \) and \( \tilde{\pi}^U \) (with \( \tilde{\pi}^D + \tilde{\pi}^U = \tilde{\pi} \)) are respectively the profit shares that matched downstream and upstream firms receive from the partnership. As will be shown later, this division is determined by bargaining between matched pairs. These four Bellman equations represent how matched and unmatched pairs obtain gains and losses from matching and dissolving in the dynamics. The first equation, for example, suggests that matched downstream pairs have (i) instantaneous gains (\( \tilde{\pi}^D \)); (ii) losses from becoming unmatched (\( V^D - \tilde{V}^D \)) that occur with a probability of \( \lambda \); and (iii) potential gains or losses from remaining matched (\( \dot{\tilde{V}}^D \)). While the first three equations are similarly interpreted, the last equation indicates that if upstream firms are matched with a potential downstream partner, they incur a relationship-specific investment \( K \) for the needs of their partner. Strictly speaking, the investment cost \( K \) must be small enough to ensure gains from becoming matched for unmatched firms (\( \tilde{\pi}^D \geq V^D \) and \( \tilde{\pi}^U - K > V^U \)), so that they have some incentives to make a costly investment. Later we explicitly derive the value functions to show the condition under which downstream firms invest in equilibrium.

In the steady state, we have \( \dot{\tilde{V}}^D = \dot{V}^D = \dot{\tilde{V}}^U = \dot{V}^U = 0 \). Setting \( \dot{\tilde{V}}^D = \dot{V}^D = 0 \) and solving for \( \tilde{V}^D \) and \( V^D \) in the Bellman equations of downstream firms yields

\[
\tilde{V}^D = \frac{\tilde{\pi}^D}{r + \lambda} + \frac{\lambda}{r + \lambda} V^D, \\
V^D = \frac{\pi}{r + \mu^D} + \frac{\mu^D}{r + \mu^D} \tilde{V}^D.
\]

In these equations, the values in the denominator \((r + \lambda)\) work as the time discount factor for matched and unmatched pairs. Regarding the value function of matched downstream firms \((\tilde{V}^D)\), for example, these firms would earn \( \tilde{\pi}^D/(r + \lambda) \) of discounted value in every period. Once they are hit by the bad shock (with a probability of \( \lambda \)) that forces them to dissolve the relationship, they would hereafter earn \( V^D \) in every period and \( V^D/(r + \lambda) \) of discounted value.

Substituting \( \tilde{V}^D \) into \( V^D \) and rearranging, we have

\[
r\tilde{V}^D = \frac{(r + \mu^D) \tilde{\pi}^D}{r + \lambda + \mu^D} + \frac{\lambda \pi}{r + \lambda + \mu^D}, \\
rV^D = \frac{(r + \lambda) \pi}{r + \lambda + \mu^D} + \frac{\mu^D \tilde{\pi}^D}{r + \lambda + \mu^D}.
\] (4)

These equations can be thought of as indicating the equilibrium proportion of \( \tilde{\pi}^D \) and \( \pi \) for matched and unmatched pairs in the downstream stage. Among matched pairs, the proportion of firms who would earn \( \tilde{\pi}^D \) in every period is \( (r + \mu^D)/(r + \lambda + \mu^D) \) and the corresponding proportion of \( \pi \) is \( \lambda/(r + \lambda + \mu^D) \).

In an analogous way, by setting \( \dot{\tilde{V}}^U = \dot{V}^U = 0 \), the Bellman equations of upstream firms are

\[
\tilde{V}^U = \frac{\tilde{\pi}^U}{r + \lambda} + \frac{\lambda}{r + \lambda} V^U, \\
V^U = \frac{\mu^U}{r + \mu^U} \tilde{V}^U - \frac{\mu^U}{r + \mu^U} K,
\]
which are in turn solved for the value functions of these firms:

\begin{align*}
    r \tilde{V}^U &= \frac{(r + \mu^U) \tilde{\pi}^U}{r + \lambda + \mu^U} - \frac{\lambda \mu^U}{r + \lambda + \mu^U} K, \\
    r \tilde{V}^U &= \frac{\mu^U \tilde{\pi}^U}{r + \lambda + \mu^U} - \frac{\mu^U (r + \lambda)}{r + \lambda + \mu^U} K.
\end{align*}

(5)

Let us now turn to bargaining on profit sharing between matched pairs. We characterize the outcome of this bargaining within matched pairs as a symmetric Nash bargaining solution where both upstream and downstream firms have equal bargaining power over ex-post gains from the relationship. The relevant utility functions for the analysis of bargaining are the downstream firm’s ex-post gains \( \tilde{V}^D - V^D \) and the upstream firm’s ex-post gains \( \tilde{V}^U - K - V^U \). Then, the Nash bargaining profit sharing uniquely solves the following maximization problem:

\[
(\tilde{\pi}^D, \tilde{\pi}^U) = \arg \max_{\tilde{\pi}^D', \tilde{\pi}^U'} \left( \tilde{V}^{D'} - V^D \right) \left( \tilde{V}^{U'} - K - V^U \right),
\]

subject to

\[\tilde{\pi}^D' + \tilde{\pi}^U' = \tilde{\pi}.
\]

Substituting (4) and (5) into the above maximization problem and solving it at \( \tilde{\pi}^D' = \tilde{\pi}^D \) and \( \tilde{\pi}^U' = \tilde{\pi}^U = \tilde{\pi} - \tilde{\pi}^D \), we have the following profit sharing rule in a stationary equilibrium (see Appendix):

\[
\begin{align*}
    \tilde{\pi}^D &= \frac{1}{2(r + \lambda) + \mu^D + \mu^U} \left[ (r + \lambda + \mu^D) \tilde{\pi} + (r + \lambda + \mu^U) \pi - (r + \lambda) (r + \lambda + \mu^D) K \right], \\
    \tilde{\pi}^U &= \frac{1}{2(r + \lambda) + \mu^D + \mu^U} \left[ (r + \lambda + \mu^U) \tilde{\pi} - (r + \lambda + \mu^U) \pi + (r + \lambda) (r + \lambda + \mu^D) K \right],
\end{align*}
\]

where \( \tilde{\pi} \) and \( \pi \) are given in (1). Using this profit sharing rule, the value functions of downstream and upstream firms in (4) and (5) are subsequently determined. Note that the relationship-specific investment cost \( K \) appears in the profit sharing rules. This reflects the fact that matched pairs maximize the joint surplus in terms of the value functions, and these functions pairs account for the future possibility of realizing a match in the dynamics: once these pairs are successfully matched with a potential partner from another side of production, they negotiate over the profit sharing rule and thereby \( K \) is split among them.

We can now check the equilibrium condition under which matched pairs have incentives to make the investment. Substituting the above profit sharing rules into the value functions, the conditions for making the investment hold \( (\tilde{V}^D > V^D \text{ and } \tilde{V}^U - K > V^U) \) if and only if

\[
\tilde{\pi} - (r + \lambda) K > \pi.
\]

(6)

Noting that \( r + \lambda \) is the time discount factor for matched firms, the investment cost \( (r + \lambda) K \) represents a fraction of the entire cost \( K \) that is paid by matched firms in every period. Thus, the left-hand represents the profit of matched pairs in each period, whereas the right-hand side represents the profit of unmatched pairs in every period (before accounting for the fixed entry
costs $F^D$ and $F^U$). Thus, the difference $\tilde{\pi} - \pi - (r + \lambda)K$ represents the surplus or economic rent created by the relationship that only matched pairs can obtain. In a stationary equilibrium where the fixed entry costs are sunk, matched pairs can earn greater profit than unmatched pairs under condition (6), which is the only reason that firms have incentives to incur the investment cost $K$.

The number of matched and unmatched firms in each stage of production is endogenously pinned down by a free entry condition that holds for each product market. (The analysis of a labor market can be omitted by Walras’ law.) In this simple dynamic model, the number of firms is determined by the equality between the value of firms’ entry and the fixed entry cost. Note that the each value function contains the four endogenous variables $(\mu^D, \mu^U, \pi, \tilde{\pi})$, and the number of firms is jointly determined by these variables. However, the probabilities of a match can be expressed in terms of $z$ from (2'), whereas the profit of matched pairs $\tilde{\pi}$ is expressed in terms of $\pi$ from (1), i.e. $\tilde{\pi} = \gamma \pi$. As a consequence, the four value functions can be expressed in terms of only the two endogenous variables $(z, \pi)$, and it is enough to consider only two value functions for free entry.

It follows from $\tilde{V}^D > V^D$ and $\tilde{V}^U - K > V^U$ that the free entry condition should hold only for unmatched pairs. Indeed, given that the fixed entry costs are the same for matched and unmatched pairs, if the free entry condition were instead imposed to matched pairs, the values of entry for unmatched pairs are negative and these firms would immediately exit. This also means that the net values of entry for matched pairs are positive, which gives enough incentives for matched pairs to make the investment cost $K$ as noted above. Hence, the free entry condition of unmatched pairs suffices to determine the number of firms:

$$\begin{cases} V^D = F^D, \\ V^U = F^U. \end{cases}$$

From the free entry conditions of unmatched pairs, the number of unmatched downstream firms $(N - n)$ and that of unmatched upstream firms $(M - n)$ are determined. Simultaneously, from the search technology, $n = \nu(M - n, N - n)$, the number of matched pairs $(n)$ is also determined. Then, these numbers jointly pin down $z = (M - n)/(N - n)$ as well as $\pi = \tilde{\pi}/\gamma$ in the model. Once these two variables are characterized in equilibrium, other endogenous variables can be written as a function of them.\footnote{While we will focus on this free entry conditions in the following analysis, it is possible to alternatively use the expected value of entry that takes account of both matched and unmatched pairs. Since the probabilities of a match for downstream and upstream firms are $\mu^D$ and $\mu^U$, and the corresponding gains are $\tilde{V}^D$ and $\tilde{V}^U - K$ respectively, the above condition in terms of the expected value is}

$$\begin{cases} \mu^D\tilde{V}^D + (1 - \mu^D)V^D = F^D, \\ \mu^U(\tilde{V}^U - K) + (1 - \mu^U)V^U = F^U. \end{cases}$$

As shown in the Appendix, the equilibrium characterizations are qualitatively similar even in this case. Intuitively, this similarity comes from the fact that the value function of unmatched pairs takes into account the future possibility of realizing a match in the dynamics. Thus, even if the free entry condition is imposed to unmatched pairs only, the equilibrium characterizations are conceptually consistent.
3.2 Equilibrium characterizations

Having described the equilibrium conditions in the model, we next move to characterizing the closed-economy equilibrium. Substituting the profit sharing rules, \( \bar{\pi}^D \) and \( \bar{\pi}^U \), into (4) and (5), the value functions of unmatched pairs can be written as follows (see Appendix for detailed derivations):

\[
\begin{align*}
 rV^D &= \pi + \frac{\mu^D [\bar{\pi} - \pi - (r + \lambda)K]}{2(r + \lambda) + \mu^D + \mu^U}, \\
 rV^U &= \mu^U [\bar{\pi} - \pi - (r + \lambda)K] \\
 &\quad + \frac{\mu^D + \mu^U}{2(r + \lambda) + \mu^D + \mu^U}.
\end{align*}
\]

To see the implication of these value functions, let us rewrite these functions as

\[
\begin{align*}
 rV^D &= \pi + \phi^D [\bar{\pi} - \pi - (r + \lambda)K], \\
 rV^U &= \phi^U [\bar{\pi} - \pi - (r + \lambda)K],
\end{align*}
\]

where

\[
\phi^D = \frac{\mu^D}{2(r + \lambda) + \mu^U + \mu^D}, \quad \phi^U = \frac{\mu^U}{2(r + \lambda) + \mu^D + \mu^U}
\]

\( \phi^D \) and \( \phi^U \) represent a conditional probability of realizing a match for downstream firms and upstream firms respectively.\(^4\) If downstream firms remain unmatched, they would earn \( \pi \) in every period (thus \( \pi \) is considered as an outside option for downstream firms); and if they find a potential partner and become matched – which occurs with probability \( \phi^D \) in the dynamics, they would earn the additional economic rent created by the partnership \( \bar{\pi} - \pi - (r + \lambda)K \). While the similar interpretation holds for upstream firms, note that the outside option for these firms is zero due to the perfect-competition assumption. In sum, these value functions represent the expected values of firms’ entry that account for both matched and unmatched possibilities in time flows.

In the long-run equilibrium where entry is unrestricted, potential firms continue to enter if there exist any opportunities of positive profit in the economy. In the current dynamic model, these opportunities are represented by the value functions derived above. Let \( V^D_e \) and \( V^U_e \) denote the net values of entry for downstream and upstream firms respectively. Then, the discounted net values of entry in each side of production are given by

\[
\begin{align*}
 rV^D_e &= r(V^D - F^D) = \pi + \phi^D [\bar{\pi} - \pi - (r + \lambda)K] - rF^D, \\
 rV^U_e &= r(V^U - F^U) = \phi^U [\bar{\pi} - \pi - (r + \lambda)K] - rF^U.
\end{align*}
\]

\(^4\)To see this, let us consider \( \phi^D \) for example. Since we use the value function of unmatched firms for free entry, these firms remain unmatched at the beginning of each period. In a search process, they have four possibilities: (i) finding a potential upstream firm with probability \( \mu^U \); (ii) failing to find a partner with probability \( r + \lambda \); (iii) being searched by an upstream firm with probability \( \mu^U \); and (iv) being failed to be searched by a partner with probability \( r + \lambda \). Given these four possibilities, the conditional probability of finding a partner for unmatched downstream firms is given by \( \phi^D \).
Clearly, \( V^D_e = 0 \) and \( V^U_e = 0 \) must hold in the stationary equilibrium. Solving these equalities for \( \pi(= \tilde{\pi}/\gamma) \) yields the following profit levels that account for the fixed costs of entry and investment:

\[
\pi = \frac{rF^D + \phi^D (r + \lambda) K}{1 + \phi^D (\gamma - 1)}, \quad (7)
\]

\[
\pi = \frac{rF^U + \phi^U (r + \lambda) K}{\phi^U (\gamma - 1)}, \quad (8)
\]

where the profit level in the left-hand side is given in (1). The number of entering firms in the downstream and upstream stages is determined by these two equalities, because these \( \pi \)'s – one for downstream firms and one for upstream firms – satisfy the free entry conditions and hence are consistent with zero expected profits.

Recall from (2') that \( \mu^D \) and \( \mu^U \) are written as a function of \( z = (M - n)/(N - n) \). It then follows from the normalized matching probability \( s(z) \) that \( \phi^D \) and \( \phi^U \) are also a function of \( z \):

\[
\phi^U(z) = \frac{s(z)}{2(r + \lambda)z + zs(z) + s(z)},
\]

\[
\phi^D(z) = \frac{zs(z)}{2(r + \lambda)z + zs(z) + s(z)}.
\]

Moreover, under the search technology with constant returns to scale, \( \phi^U \) is strictly decreasing in \( z \) whereas \( \phi^D \) is strictly increasing in it. Intuitively, this property reflects the fact that these variables are the conditional probabilities of being matched for downstream and upstream firms. The larger is \( z \), the larger is the number of unmatched upstream firms \( M - n \) relative to that of unmatched downstream firms \( N - n \). Downstream firms have a better chance of finding partners in another side of production and thus \( \phi^D \) increases with \( z \). This at the same time also implies that upstream firms have a worse chance of a match and hence \( \phi^U \) declines with \( z \).

Using this property, Figure 2 depicts the equilibrium conditions, (7) and (8), in \((z, \pi)\)-space. For the \(UU\) curve that represents (8), this curve is monotonically increasing in \( z \). For the \(DD\)
curve that represents (7), on the other hand, this curve is monotonically decreasing in \(z\) if
\[(r + \lambda)K < rF^D(\gamma - 1).\]

Substituting (7) into (6), it is easily verified that this condition is equivalent to (6). Thus, the \(DD\) curve is monotonically decreasing in \(z\) as long as matched pairs have incentives to invest for the relationship. The economic mechanism behind Figure 2 is as follows. For \textit{downstream} firms, the larger \(z\) implies the higher possibility of a match \(\mu^D(z) = s(z)\) and higher expected profits. Since this induces a larger number of entrants and more intense competition in the downstream stage, the profitability of downstream firms subsequently declines and the \(DD\) curve decreases with \(z\). For \textit{upstream} firms, the larger \(z\) implies the lower probability of a match \(\mu^U = s(z)/z\) and lower expected profits. Since this induces a smaller number of entrants and less intense competition in the upstream stage, the profitability of upstream firms subsequently improves and the \(UU\) curve increases with \(z\). These features of the \(DD\) and \(UU\) curves ensure the existence and uniqueness of the stationary equilibrium in the closed economy, and the intersection of these curves determines the equilibrium variables \(z\) and \(\pi\), as represented in the figure.

Once \(z\) and \(\pi\) are uniquely determined, other endogenous variables \((\mu^D, \mu^U, \tilde{\pi}, n, M, N)\) in the steady state can be written as a function of only these two variables. The probabilities of a match are \(\mu^D = s(z)\) and \(\mu^U = s(z)/z\) and the profit of matched pairs is \(\tilde{\pi} = \gamma \pi\). The number of matched pairs, \(n\), is recovered from rearranging (1) and using (3):

\[
\pi = \frac{E}{\gamma \sigma n \left[1 + \frac{1}{\gamma} \left(\frac{N-n}{n}\right)\right]} \iff n = \frac{E}{\gamma \sigma \pi \left[1 + \frac{\lambda}{s(z)}\right]}. \tag{9}
\]

The total numbers of downstream and upstream firms are then determined by (2’) and (3):

\[
M = \left[\frac{z\lambda + s(z)}{s(z)}\right] n, \quad N = \left[\frac{\lambda + s(z)}{s(z)}\right] n.
\]

Finally, the price index \(P\) is determined by \(n\) and \(N\), which completes the characterization of the unique stationary equilibrium in the closed-economy model.

### 3.3 Relationship specificity and market thickness

Building on the above equilibrium characterizations, we next examine the impact of relationship specificity \(\gamma\) on the equilibrium variables. More specifically, we consider comparative statics with respect to \(\gamma\) to see how \(z\) and \(\pi\) are affected by \(\gamma\). (Recall that \(\gamma\) is sector-specific and it varies across sectors.) To compare two sectors with different \(\gamma\)’s, we add primes (‘) to relevant variables associated with higher relationship specificity (\(\gamma’ > \gamma\)). Though we focus on \(\gamma\) here, comparative statics with respect to the other exogenous variables are similarly conducted (see Appendix).

A simple inspection of (7) and (8) reveals that an increase in \(\gamma\) shifts the \(DD\) and \(UU\) curves down and the profit of unmatched pairs \(\pi\) decreases with \(\gamma\) (\(\pi’ < \pi\)), while the profit of matched pairs \(\tilde{\pi}\) increases (\(\tilde{\pi’} > \tilde{\pi}\)). On the other hand, the ratio of unmatched pairs \(z\) increases with \(\gamma\).
(z' > z). Indeed, from the free entry conditions, (7) and (8), it follows that

\[ \pi + \phi^D [\tilde{\pi} - \pi - (r + \lambda)K] = rF^D, \quad (7') \]

\[ \phi^U [\tilde{\pi} - \pi - (r + \lambda)K] = rF^U, \quad (8') \]

where the left-hand side is the per-period expected profits (after accounting for the investment cost) and the right-hand side is the discounted entry cost. Canceling out \( \tilde{\pi} - \pi - (r + \lambda)K \) from both equations and using \( z = \frac{\phi^D}{\phi^U} \), we have

\[ z = \frac{rF^D - \pi}{rF^U}. \]

Since \( \pi \) declines with \( \gamma \), \( z \) should increase with \( \gamma \) in order to keep the free entry condition in both sides of production. From (9), then, the number of matched pairs accordingly increases \( n' > n \), whereas the ratio of unmatched firms to matched firms changes as \( n'/(N' - n') > n/(N - n) \) in the downstream stage and \( n'/(M' - n') < n/(M - n) \) in the upstream stage.

To develop the intuition behind this result, it is useful to notice (7') and (8'). The increase in \( \gamma \) has two opposing effects on the expected profits. First, the economic rent created by the relationship, \( \tilde{\pi} - \pi - (r + \lambda)K \), is increasing in \( \gamma \) and the gains from being matched are greater in a higher \( \gamma \) sector. Second, the outside option of downstream firms, \( \pi \), is decreasing in \( \gamma \) and their bargaining position is worse in a higher \( \gamma \) sector. It is easy to see that, for upstream firms, this second effect is absent and thus the higher \( \gamma \) leads to the higher expected profits and induces a larger number of entrants. The higher is \( \gamma \), the more intense is competition and the lower is \( \pi \) in the upstream stage, and consequently the \( UU \) curve shifts down as \( \gamma \) increases. For downstream firms, on the other hand, though the economic rent increases with \( \gamma \), the outside option simultaneously declines and the expected profits increases relatively less sharply than those for upstream firms. Thus, though a higher \( \gamma \) induces a larger number of entrants in the downstream stage as well, its increase is less proportional to that in the downstream stage. In Figure 2, this is captured by the fact that the downward shift in the \( DD \) curve is less than that in the \( UU \) curve, which leads to an increase in \( z \) in equilibrium.

It is now possible to see how the market thickness is affected by relationship specificity \( \gamma \). Let \( X \equiv (N - n)x \) denote the aggregate output supplied through markets by unmatched pairs, and \( \tilde{X} \equiv n\tilde{x} \) the aggregate output transacted within matched pairs. Then, using the equilibrium characterizations derived above, these are expressed as

\[ X = \frac{\sigma - 1}{\sigma c} \frac{E}{1 + \frac{\gamma s(z)}{\lambda}}, \quad \tilde{X} = \frac{\sigma - 1}{\sigma c} \frac{E}{1 + \frac{X}{\gamma s(z)}}. \]

We define the “market thickness” as the ratio of aggregate output supplied through markets \( X \) to aggregate output transacted through within pairs \( \tilde{X} \); that is, the market is thicker if relatively more products are supplied by unmatched pairs through markets (than by matched pairs within their relationships). Since one unit of final good requires one unit of component in this model, these variables can be also regarded as the aggregate input used for unmatched and matched
pairs. It is then immediate to show that the ratio of these aggregates are decomposed into

$$\frac{X}{\tilde{X}} = \frac{N - n}{n} \cdot \frac{x}{\tilde{x}}$$

$$= \frac{\lambda}{s(z)} \cdot \frac{1}{\gamma},$$

where \((N - n)/n = \lambda/s(z)\) is the relative number of unmatched pairs (extensive margin) and

\[x/\tilde{x} = 1/\gamma\]

is the relative output level of these pairs (intensive margin). Note that the ratio of the intensive margin is necessarily smaller than one, whereas the ratio of the extensive margin can be greater or smaller than one. In addition, not only does the intensive margin, but the extensive margin also decreases with \(\gamma\), because \(z\) is increasing in \(\gamma\) as shown above. Therefore, our model suggests that the higher is relationship specificity, the thinner is the market, which occurs through both intensive and extensive margins. This observation is summarized in the following proposition.

**Proposition 1.** The market thickness \(X_j/\tilde{X}_j\) in sector \(j\) is lower, the higher is the relationship specificity \(\gamma_j\).

Our model provides a possible theoretical explanation for Nunn’s (2007) empirical evidence on export patterns and relationship specificity. To investigate this relationship, Nunn measures the proportion of components that are sold in the market and components that are transacted within firms across various sectors. Using this measure, he argues that if this proportion (i.e. market thickness) varies with relationship specificity, countries with better contract enforcement would have a comparative advantage in sectors for which relationship-specific investments are more important. However, his analysis does not deeply explain why the market thickness varies with relationship specificity. The current model clearly shows how the market thickness \((X/\tilde{X})\) is endogenously determined by the relationship specificity \((\gamma)\) by affecting the matching possibility \((s(z))\) among upstream and downstream firms in the dynamics, simultaneously pinning down the number of matched and unmatched pairs \((n, M, N)\).

So far, we have been concerned with the characterization of the closed-economy equilibrium. Our main interest in this paper is however in understanding whether an increase in final-good trade stimulates offshoring and component trade in a mutually complementary manner, a key question that will be addressed in the next section.

### 4 Open-economy equilibrium

This section investigates a global economy in which two symmetric countries that are previously described engage in final-good trade. To export final products to another country, firms incur iceberg transport cost \(\tau\) and fixed export-entry cost \(F_x\). We assume that \(\tau\) and \(F_x\) fall in a range such that only the goods with high-quality can be profitably exported (The exact condition will be shown later); firms export their products if and only if they are matched with their individual upstream firms.
Let us first consider the optimal firm behavior in each period. The pricing rules of matched and unmatched pairs in their domestic market are the same as before, \( p = \bar{p}_d = \sigma c / (\sigma - 1) \), as two countries share the same wage rate, which is normalized to one. By the above assumption, only matched pairs can export with a higher pricing rule that accounts for the transport cost: \( \bar{p}_x = \tau \sigma c / (\sigma - 1) = \tau p \). Then, the price index \( P \) in each country is written as

\[
P = \left[ n\bar{p}^{1-\sigma} + (N-n)\frac{p_1^{1-\sigma}}{\gamma} + n\bar{p}^{1-\sigma}_x \right]^{1/\sigma} = \frac{\sigma c}{\sigma - 1} \left[ n(1 + \tau^{1-\sigma}) + \frac{N-n}{\gamma} \right]^{1/\sigma}.
\]

Note that the number of matched and unmatched pairs in the open economy is typically different from that in the closed economy. Using this price index, we have the equilibrium output and profit levels in the domestic market for matched and unmatched pairs:

\[
x = \sigma - 1 \frac{E}{\sigma c \gamma \left[ n(1 + \tau^{1-\sigma}) + \frac{N-n}{\gamma} \right]}, \quad \pi = \frac{E}{\gamma \sigma \left[ n(1 + \tau^{1-\sigma}) + \frac{N-n}{\gamma} \right]},
\]

\[
\bar{x}_d = \sigma - 1 \frac{E}{\sigma c \left[ n(1 + \tau^{1-\sigma}) + \frac{N-n}{\gamma} \right]}, \quad \bar{\pi}_d = \frac{E}{\sigma \left[ n(1 + \tau^{1-\sigma}) + \frac{N-n}{\gamma} \right]}.
\]

The export output and revenue levels of matched pairs are respectively \( \bar{x}_x = \tau^{-\sigma} \bar{x}_d \) and \( \bar{x}_x = \tau^{1-\sigma} \bar{\pi}_d \), both of which are smaller than \( x_d \) and \( \bar{\pi}_d \) reflecting the transport cost. Letting \( \bar{x} = \bar{x}_d + \bar{x}_x = (1 + \tau^{-\sigma}) \bar{x}_d \) and \( \bar{\pi} = \bar{\pi}_d + \bar{\pi}_x = (1 + \tau^{1-\sigma}) \bar{\pi}_d \) represent the total output and profit levels of matched pairs, the differences between matched and unmatched pairs become bigger in the open economy than those in the closed economy:

\[
\frac{\bar{x}}{x} = (1 + \tau^{-\sigma}) \gamma > \frac{\bar{x}_a}{x_a} = \gamma, \quad \frac{\bar{\pi}}{\pi} = (1 + \tau^{1-\sigma}) \gamma > \frac{\bar{\pi}_a}{\pi_a} = \gamma,
\]

where subscript \( a \) stands for autarky.

### 4.1 Impact of trade

Following the previous section, we can solve for the equilibrium characterizations of the downstream and upstream stages in the open-economy setting. Since the equilibrium conditions are similar, we relegate detailed derivations to the Appendix. In particular, the equilibrium profits of downstream and upstream firms that are consistent with zero expected profits are given by setting the net values of entry for unmatched pairs to be zero \((V^D = F^D \text{ and } V^U = F^U)\):\(^5\)

\[
\pi = \frac{r F^D + \phi^D (r + \lambda) (F_x + K)}{1 + \phi^D [(1 + \tau^{1-\sigma}) \gamma - 1]}, \quad (10)
\]

\[
\pi = \frac{r F^U + \phi^U (r + \lambda) (F_x + K)}{\phi^U [(1 + \tau^{1-\sigma}) \gamma - 1]}, \quad (11)
\]

\(^5\)The condition can be alternatively expressed in terms of the expected values: \( \mu^D (V^D - F_x) + (1 - \mu^D) V^D = F^D \) and \( \mu^U (V^U - K) + (1 - \mu^U) V^U = F^U \). In either case, the equilibrium characterizations are qualitatively similar (see the Appendix for proof).

15
where $\phi^D$ and $\phi^U$ are exactly the same as those in the closed economy. It is easy to see that, as $\tau \to \infty$ and $F_x \to 0$, (10) and (11) respectively converge to (7) and (8). These two $\pi$’s are also equivalent with the following equalities:

\[
\begin{align*}
\pi + \phi^D [\tilde{\pi} - \pi - (r + \lambda)(F_x + K)] &= rF^D, \\
\phi^U [\tilde{\pi} - \pi - (r + \lambda)(F_x + K)] &= rF^U,
\end{align*}
\]

where the left-hand side is the per-period expected profits (after accounting for the investment cost and export-entry cost) and the right-hand side is the discounted entry cost. In the stationary equilibrium, these equalities jointly pin down the number of matched and unmatched pairs in both sides of production.

Since the trade costs are exogenous variables, (10) and (11) can be similarly drawn in $(z, \pi)$-space as in the closed economy. In particular, the $UU$ curve is monotonically increasing in $z$ while the $DD$ curve is monotonically decreasing in $z$ if and only if the economic rent generated by the relationship after opening trade, $\tilde{\pi} - \pi - (r + \lambda)(F_x + K)$, is positive:

\[
\tilde{\pi} - (r + \lambda)(F_x + K) > \pi.
\]

(12)

In (12), the left-hand side is the per-period joint profit for matched pairs in the open economy, and the right-hand side is the per-period outside option for these pairs. Under this condition, we have $\tilde{V}^D - F_x > V^D$ and matched downstream firms have enough incentives to invest for $F_x$; similarly, $\tilde{V}^U - K > V^U$ and matched upstream firms have enough incentives to invest for $K$. From these features of the $DD$ and $UU$ curves, then, it immediately follows that there exists a unique open-economy equilibrium.

The question that remains is how the equilibrium variables are affected by final-good trade in the presence of trade costs. To consider this, we must invoke the assumption that only matched pairs can sell their final goods and earn sufficient profits to cover trade costs. For this condition to be internally consistent with the current setup, the profits of matched and unmatched pairs must satisfy, in equilibrium,

\[
(1 + \tau^{1-\sigma})\gamma \pi - (r + \lambda)(F_x + K) > \gamma \pi - (r + \lambda)K,
\]

\[
(1 + \tau^{1-\sigma})\pi - (r + \lambda)F_x < \pi.
\]

(13)

The first equality indicates that matched pairs can earn a higher per-period profit by exporting to the foreign market than by serving only the domestic market. Similarly, the second equality indicates that unmatched pairs can earn a higher per-period profit by serving only the domestic market than by exporting for the foreign market. These are the exact conditions imposed on $\tau$ and $F_x$ under which only matched pairs can profitably export.

Note that condition (12) is subsumed by the first condition in (13) since the left-hand side is the same and $\gamma \pi - (r + \lambda)K > \pi$ is assumed to hold (otherwise matched firms have no incentive to make the relationship-specific investment $K$). In other words, if the first inequality in (13) holds,
then (12) automatically holds. Hence (13) suffices to characterize the open-economy equilibrium in which only matched pairs are able to export. The two inequalities in (13) simultaneously hold if and only if

\[
\frac{(r + \lambda)F_x}{\gamma \tau^{1-\sigma}} < \pi < \frac{(r + \lambda)F_x}{\tau^{1-\sigma}}.
\]

In the equilibrium, \(\pi\) should not be sufficiently high because this would induce unmatched pairs to export by raising their expected profit; at the same time, \(\pi\) should not also be sufficiently low because this would prevent matched pairs from incurring the fixed export-entry cost through the future possibility of a match. Substituting (10) and (11) into the above inequality, the trade costs have to satisfy

\[
\left[1 + \phi D(\gamma - 1)(r + \lambda)^{1-\sigma}\right] \frac{(r + \lambda)F_x}{\gamma \tau^{1-\sigma}} < rF_D + \phi D(r + \lambda)K < \left[1 + \phi U(\gamma - 1)(1 + \tau^{1-\sigma})(r + \lambda)\right] \frac{(r + \lambda)F_x}{\tau^{1-\sigma}},
\]

where the first and second restrictions respectively apply for the \(DD\) and \(UU\) curves.

Under the circumstance, we can readily explore the impact of trade on the two equilibrium variables, \(z\) and \(\pi\). Figure 3 illustrates the closed- and open-economy equilibria at points \(A\) and \(E\) respectively. Comparing (7) and (10) for the \(DD\) curve or (8) and (11) for the \(UU\) curve, we find that both curves shift down in the open-economy equilibrium and, from the intersection of these two curves, the profit of unmatched pairs declines after opening trade \((\pi < \pi_a)\). The profit of matched pairs simultaneously increases \((\tilde{\pi} > \tilde{\pi}_a)\).\(^7\) Intuitively, this downward shift in both curves occurs though the increase of the economic rent in the open-economy equilibrium, i.e., \(\tilde{\pi} - \pi - (r + \lambda)(F_x + K) > \tilde{\pi}_a - \pi_a - (r + \lambda)K\); since the export sales contribute to an increase in this rent, the expected profit of being matched rises for both upstream and downstream firms. This encourages a large number of entrants in both sides of production, which in turn reduces

\[\text{Figure 3 – Equilibrium in the open economy}\]
the equilibrium π’s that are consistent with zero expected profits. Further, this downward shift in π is smaller for the DD curve due to the decline of the outside option of downstream firms. In the above figure, this reflects the fact that the expected profit of downstream firms at B is greater than that of upstream firms at C. Because there are a greater profit opportunity (and hence a greater number of entrants) in the downstream stage at za, this subsequently reduces the likelihood of a match and expected profit for downstream firms and, at the end, the number of downstream firms declines. The opposite is true for the upstream stage and the number of upstream firms rises. In equilibrium, this stabilizing force through the matching possibilities increases z. Opening trade thus gives rise to not only a profit-allocation effect toward matched pairs (π < πa and ˜π > ˜πa), but also a composition effect among unmatched pairs (z > za).

Using this impact on the equilibrium characterizations, we can see how the other equilibrium variables are effected by opening trade. In particular, it is worth mentioning the equilibrium number of matched pairs, which is given by

\[ n = \frac{E}{\gamma \sigma \pi \left( 1 + \tau^1 - \sigma \right) + \frac{\lambda}{s(z)}}. \]  

(14)

Comparing (9) and (14), we find that there are opposing effects on the number of matched pairs n. On one hand, trade invites foreign competitor and freer trade (in terms of smaller τ) leads to a smaller n. On the other hand, trade gives rise to a greater profit opportunity and a better chance of finding a partner for matched pairs, and freer trade (in terms of smaller π or a larger s(z)) leads to a larger n. Though whether n increases or not is indeterministic, the total number of matched pairs, (1 + τ1 - σ)n, is larger in the open economy. Since \( E = \beta L \) holds before/after trade in general equilibrium, it follows from πa > π and s(z) > s(za) that

\[ (1 + \tau^1 - \sigma)n - na = (1 + \tau^1 - \sigma)(\pi_a - \pi) + \frac{\lambda}{\gamma} \left[ \frac{(1 + \tau^1 - \sigma)\pi_a}{s(za)} - \frac{\pi}{s(z)} \right] > 0. \]

Furthermore, the share of unmatched pairs in the economy becomes lower relative to autarky: \( (N - n)/n(1 + \tau^1 - \sigma) < (N_a - na)/na \). Thus, on top of the profit reallocation effect toward matched pairs, the increase in the number of high-quality varieties produced by these pairs is also the welfare gain from trade in the current model. Indeed, as is well-known in the empirical literature, the rise in high-quality products that are internationally traded is of particular significance for welfare estimations.\(^8\)

Regarding the market thickness, we have in the open economy that

\[ \frac{X}{\tilde{X}} = \frac{\lambda}{s(z)} \cdot \frac{1}{(1 + \tau^1 - \sigma)\gamma}. \]

It is clear that opening trade leads to the thinner market through the two margins: the intensive margin \( x/\tilde{x} = 1/(1 + \tau^1 - \sigma)\gamma \) decreases due to imports of matched pairs from a foreign country, while the extensive margin \( (N - n)/n = \lambda/s(z) \) decreases due to an increased probability of a

\(^{8}\)There is a growing body of evidence that goods traded internationally are on average of higher quality than those sold domestically. See for example Hummels and Skiba (2004).
match for downstream firms. Therefore, our model predicts that the market becomes thinner in the open economy relative to autarky. This finding is summarized in the following proposition:

**Proposition 2.** In any sector \( j \in J \), the market thickness \( X_j / \tilde{X}_j \) becomes lower after opening trade in final goods.

This proposition reflects to some extent our assumption that only matched pairs can export, which decreases the relative output level of unmatched pairs (intensive margin). Our model also shows, however, that the market becomes thinner through the decline in the relative number of unmatched pairs (extensive margin) as well. Recent empirical evidence suggests the importance of the extensive margin for understanding the gains from trade (e.g., Amiti and Konings, 2007; Goldberg et al., 2010). Though these studies focus on the extensive margin of imported input from a foreign country, our model instead focuses on the extensive margin of domestic input in a home country. By so doing, we find that the extensive margin matters even without input trade, since final-good trade affects the profitability in the vertically related markets and thereby it improves the probability of realizing a match for downstream firms. This stabilizing influence that arises from searching and matching leads to the thinner market in the model.

### 4.2 Impact of trade liberalization

Up until now we have examined the drastic change that the economy would experience when it suddenly moves from autarky to trade. Next, we investigate effects of a gradual reduction of trade cost (in the form of trade liberalization) on the equilibrium variables. More specifically, we conduct comparative statics with respect to \( \tau \) and \( F_x \) to see how \( z \) and \( \tilde{\pi} \) are affected by these parameters in the steady state.

Recall first that (10) and (11) respectively converge to (7) and (8) if \( \tau \to \infty \) and \( F_x \to 0 \). This implies that the reduction of the two trade costs works for the equilibrium variables in the opposite direction; that is, the reduction of transport cost is captured by a decrease of \( \tau \) while the reduction of fixed export-entry cost is captured by an increase of \( F_x \) in the current model. Hence, when primes are added to all variables after the reduction of trade costs, we must have \( \tau' < \tau \) and \( F_x' > F_x \).

Keeping these differences in mind, let us first consider the reduction of \( \tau \) for a fixed \( F_x \). A simple inspection of (10) and (11) reveals that, as \( \tau \) falls, the DD and UU curves shift down, and thus the reduction of \( \tau \) entails a decrease in \( \pi \); it simultaneously raises the profit of matched pairs \( \tilde{\pi} \). From condition (12), it also follows that the reduction in \( \tau \) side leads to an increase in \( z \). Therefore, the reduction of \( \tau \) results in \( z' > z \) and \( \pi' < \pi \) (\( \tilde{\pi'} > \tilde{\pi} \)). The intuition behind this result is similar with the increase in \( \gamma \) in the closed economy; that is, the reduction in \( \tau \) raises the economic rent \( \tilde{\pi} - \pi - (r + \lambda)(F_x + K) \) and thus the expected profit of being matched. Since this induces a larger number of entrants, there arises more intense competition in each side of production and the two \( \pi \)’s that are consistent with zero expected profits decrease. As a result, the DD and UU curves shift down.

As for the increase of \( F_x \), it is easily seen from (10) and (11) that both DD and UU curves shift up and thereby \( \pi \) rises. This also leads to an increase in the profit of matched pairs, which
is more proportional to an increase in $\pi$ ($\tilde{\pi}' > \pi'$). Moreover, it is easily verified that $z$ decreases with $F_x$. Therefore, the increase of $F_x$ results in $z' < z$ and $\pi' > \pi$ ($\tilde{\pi}' > \tilde{\pi}$). The reason for this is opposite to the reduction of $\tau$ explained above; that is, the increase in $F_x$ decreases the economic rent and thus the expected profits of being matched. Since this induces a smaller number of entrants, there arises less intense competition in each side of production and the two $\pi$’s that are consistent with zero expected profits increase. As a result, the $DD$ and $UU$ curves shift up.

Based on these comparative statics, we can now examine how differently trade liberalization affects the equilibrium across sectors. In particular, we are interested in whether or not the reduction of trade cost magnifies the effect of relationship specificity $\gamma$ on $z$ and $\pi$. From (10) and (11), it follows that the impact from the reduction of trade costs on $z$ and $\pi$ is greater, the higher is the relationship specificity $\gamma$. This suggests that the gains from trade – in terms of profit allocations toward matched pairs – are greater in sectors requiring higher relationship-specific investments, because the firms’ profits are more significantly allocated to matched pairs via a change in $z$ in such sectors.

Finally, we conclude the analysis by applying this result to the market thickness $X/\tilde{X}$. The reduction of $\tau$ leads to an decrease in both extensive and intensive margins and hence the market thickness $X/\tilde{X}$. By contrast, the reduction of $F_x$ has no impact on the intensive margin while the extensive margin increases, which in total raising the market thickness. Thus, the model predicts that the reduction of trade cost have two different channels on the market thickness: as variable trade costs declines, the volume of trade supplied through the market will decrease relatively more than that transacted within firms, whereas the opposite is true for the decline of fixed trade costs. Moreover, this impact is larger, the higher is $\gamma$ – which occurs through both the extensive and intensive margins.

**Proposition 3.** When variable trade costs decline, the market becomes thinner as the world is more integrated. Moreover, this effect is larger in a sector with a higher relationship specificity. When fixed trade costs decline, the opposite is true for the market thickness.

## 5 Conclusion

This paper has investigated (i) the relationship between the relationship specificity of investment as an industry characteristic and the market thickness, and (ii) the impact of trade on the market thickness. We find that (i) the higher is the relationship specificity, the thinner is the market; and (ii) opening trade (or reduction of trade costs) leads to a thinner market.

We plan to analyze the effect of offshoring of the component production on international trade in final goods. Does offshoring encourage trade in final goods? Does an increase in trade in final goods, in turn, increase offshoring? We shall obtain answers to these questions and further insights in the interaction between offshoring and international trade.

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9More specifically, we have $\partial^2 z / \partial \tau \partial \gamma > 0$, $\partial^2 z / \partial F_x \partial \gamma > 0$, $\partial^2 \pi / \partial \tau \partial \gamma < 0$ and $\partial^2 \pi / \partial F_x \partial \gamma < 0$. To prove these cross-partial derivatives, we need $\phi^{\nu''}(z) > 0$ and $\phi^{\nu''}(z) < 0$, which in turn comes from the concavity of the search technology $\nu(\cdot, \cdot)$ (see the Appendix for details).
References


Appendix

A Proofs of the closed-economy equilibrium

A.1 Proof of the bargaining solution for matched pairs

The bargaining problem within matched pairs is

$$\max_{\hat{\pi}^D, \hat{\pi}^U} \left( \hat{V}^D - r \hat{V}^U \right) \left( \hat{V}^U - K - V^U \right),$$

subject to $\hat{\pi}^D + \hat{\pi}^U = \hat{\pi}$. Substituting the Bellman equations in each stage of production ($\hat{V}^D = \frac{\hat{\pi}^D}{r + \lambda} + \frac{\lambda}{r + \lambda} V^D$ and $\hat{V}^U = \frac{\hat{\pi}^U}{r + \lambda} + \frac{\lambda}{r + \lambda} V^U$) into the maximization problem gives

$$\max \left( \frac{\hat{\pi}^D}{r + \lambda} - r \hat{V}^D \right) \left( \frac{\hat{\pi} - \hat{\pi}^D}{r + \lambda} - r \hat{V}^U - \frac{\lambda}{r + \lambda} - K \right)$$

$$\iff \max \frac{1}{r + \lambda} \left( \frac{\hat{\pi}^D - r \hat{V}^D}{r + \lambda} \right) \left( \frac{\hat{\pi} - \hat{\pi}^D}{r + \lambda} - r \hat{V}^U - \frac{\lambda}{r + \lambda} - K \right).$$

Solving this problem, we have

$$\hat{\pi}^D = \frac{1}{2} \left( \frac{\hat{\pi} + r \left( V^D - V^U \right) - (r + \lambda)K}{r + \lambda + \mu^U} \right),$$
$$\hat{\pi}^U = \frac{1}{2} \left( \frac{\hat{\pi} + r \left( V^U - V^D \right) + (r + \lambda)K}{r + \lambda + \mu^U} \right).$$

The result follows from noting $\hat{\pi}^D = \hat{\pi}^D$ and $\hat{\pi}^U = \hat{\pi}^U = \hat{\pi} - \hat{\pi}^D$ and substituting (4) and (5) into the above.

A.2 Proof of the value functions for unmatched pairs

From (4) and (5), rewrite the value functions of unmatched pairs as

$$r \hat{V}^D = \pi + \frac{\mu^D (\hat{\pi}^D - \pi)}{r + \lambda + \mu^D},$$
$$r \hat{V}^U = \frac{\mu^U (\hat{\pi}^U - (r + \lambda)K)}{r + \lambda + \mu^U}.$$ From the profit sharing rules, we have

$$\hat{\pi}^D - \pi = \frac{(r + \lambda + \mu^D) [\hat{\pi} - \pi - (r + \lambda)K]}{2(r + \lambda + \mu^D + \mu^U)},$$
$$\hat{\pi}^U - (r + \lambda)K = \frac{(r + \lambda + \mu^U) [\hat{\pi} - \pi - (r + \lambda)K]}{2(r + \lambda + \mu^D + \mu^U)}.$$

The interpretation of these equations is as follows. For matched downstream firms, they have the outside option $\pi$ in every period; besides, if they continue the relationship, they earn additional economic rents $\hat{\pi} - \pi - (r + \lambda)K$ in every period. If they are hit by a bad shock that occurs with probability $\lambda$, then they can find a partner with probability $r + \mu^D$ and thus the conditional probability of remaining matched is $(r + \lambda + \mu^D)/(2(r + \lambda + \mu^D + \mu^U))$. Substituting these values into the the value functions gives the results.

A.3 Proof of the free entry condition with expected values

We show that, even if the free entry conditions are represented in terms of expected values, the equilibrium characterizations are qualitatively similar with that in the main text.

Consider the free entry conditions with the expected values, which are defined as

$$\mu^D r \hat{V}^D + (1 - \mu^D) r V^D = r F^D,$$
$$\mu^U r (\hat{V}^U - K) + (1 - \mu^U) r V^U = r F^U.$$
For this formulation, we need the explicit values of $V^D$ and $V^U - K$. Rewrite the value functions of matched pairs as
\[
V^D = \pi + \frac{(r + \mu^D)(\hat{\pi} - \pi - (r + \lambda)K)}{r + \lambda + \mu^D},
\]
\[
V^U - K = \frac{(r + \mu^U)(\hat{\pi} - \pi - (r + \lambda)K)}{r + \lambda + \mu^U}.
\]
Substituting (A.1) into the above equations gives
\[
V^D = \pi + \frac{(r + \mu^D)[\hat{\pi} - \pi - (r + \lambda)K]}{2(r + \lambda) + \mu^D + \mu^U},
\]
\[
V^U - K = \frac{(r + \mu^U)[\hat{\pi} - \pi - (r + \lambda)K]}{2(r + \lambda) + \mu^D + \mu^U}.
\]
Using $V^D$ and $V^U - K$ derived above and $V^D$ and $V^U$ derived from Appendix A.2, the expected values of entry are then given by
\[
\mu^D V^D + (1 - \mu^D) V^D = (1 + r) \left\{ \pi + \phi^D [\hat{\pi} - \pi - (r + \lambda)K] \right\}
\]
\[
= (1 + r) V^D,
\]
\[
\mu^U V^U - K + (1 - \mu^U) V^U = (1 + r) \left\{ \phi^U [\hat{\pi} - \pi - (r + \lambda)K] \right\}
\]
\[
= (1 + r) V^U.
\]
Thus, the expected value of entry is proportional to the value of unmatched pairs. Setting $\mu^D V^D + (1 - \mu^D) V^D = rV^D$ and $\mu^U V^U - K + (1 - \mu^U) V^U = rV^U$, and solving for $\hat{\pi}$ yields
\[
\hat{\pi} = \frac{\gamma [rF^D + \phi^D (r + \lambda)K]}{1 + \phi^D (\gamma - 1)},
\]
\[
\hat{\pi} = \frac{\gamma [rF^U + \phi^U (r + \lambda)K]}{\phi^U (\gamma - 1)},
\]
where $\hat{\phi}^D \equiv (1 + r)\phi^D$ and $\hat{\phi}^U \equiv (1 + r)\phi^U$. It is clear that these two $\hat{\pi}$’s are analogous to (7) and (8) in the main text. Further, since the discount factor $r$ is an exogenous variable, we can easily check that the $UU$ curve is upward-sloping whereas the $DD$ curve is downward-sloping in $(z, \hat{\pi})$-space if condition (6) or equivalently condition (6’) holds. These observations jointly prove the existence and uniqueness of the closed-economy equilibrium and the analysis in section 3 is qualitatively held even under the expected values of entry.

A.4 Proof of comparative statics

We first show that $\partial z / \partial \gamma > 0$. Combining (7) with (8) and rearranging,
\[
(r(\phi^U F^D - \phi^D F^U))(\gamma - 1) = rF^U + \phi^U (r + \lambda)K.
\]
Note that the since right-hand side is positive, so is the left-hand side, i.e., $\phi^U F^D - \phi^D F^U > 0$. Then, differentiating the above equality with respect to $\gamma$ (with $K'(\gamma) = 0$) yields
\[
\frac{\partial z}{\partial \gamma} = -\frac{r(\phi^U F^D - \phi^D F^U)}{\phi^U (\gamma - 1) rF^D - (r + \lambda)K}.
\]
The result follows from noting that $\mu^D(z) > 0$, $\mu^U(z) < 0$, $\phi^U F^D - \phi^D F^U > 0$ and $(\gamma - 1) rF^D - (r + \lambda)K > 0$ (under condition (6) or (6’)). From Figure 2, we also know that $\partial \pi / \gamma < 0$ ($\partial \hat{\pi} / \gamma > 0$). Applying these results, comparative statics on other equilibrium variables are readily examined:
\[
\frac{\partial \hat{X}}{\partial \gamma} > 0, \quad \frac{\partial X}{\partial \gamma} < 0, \quad \frac{\partial (X/\hat{X})}{\partial \gamma} < 0, \quad \frac{\partial m}{\partial \gamma} > 0, \quad \frac{\partial M}{\partial \gamma} > 0, \quad \frac{\partial N}{\partial \gamma} > 0.
\]
The above results are derived under condition $K'(\gamma) = 0$ for simplicity. If $K'(\gamma) > 0$, we need $\dot{\pi} - (r + \lambda)K$ is increasing in $\gamma$. To see why, it follows from (8) that

$$\frac{\partial z}{\partial \gamma} = -\frac{\phi'(z)}{\phi'(z)(\gamma - 1)\pi - (r + \lambda)K},$$

where the denominator of the right-hand side is negative under (6) and $\mu'(z) < 0$. Hence, $\partial z/\gamma > 0$ if and only if $\frac{\partial z}{\gamma} - (r + \lambda)\frac{\partial K}{\partial \gamma} > 0$. It follows from $\frac{\partial z}{\gamma} < 0$ that the sufficient condition is $\frac{\partial z}{\gamma} - (r + \lambda)\frac{\partial K}{\partial \gamma} > 0$.

Following the same steps, we can examine comparative statics on other exogenous variables in the model $(\lambda, K, F^D, F^U)$. Since its derivation is similar, we report only the results below:

$$\frac{\partial \dot{\pi}^D}{\partial R} > 0, \frac{\partial \pi^D}{\partial R} > 0, \frac{\partial \ddot{\pi}^D}{\partial R} > 0, \frac{\partial \dot{\pi}^D}{\partial \lambda} > 0, \frac{\partial \pi^D}{\partial \lambda} > 0, \frac{\partial \ddot{\pi}^D}{\partial \lambda} > 0, \frac{\partial \dot{\pi}^D}{\partial M} > 0, \frac{\partial \pi^D}{\partial M} > 0, \frac{\partial \ddot{\pi}^D}{\partial M} > 0, \frac{\partial \dot{\pi}^D}{\partial \beta} > 0, \frac{\partial \pi^D}{\partial \beta} > 0, \frac{\partial \ddot{\pi}^D}{\partial \beta} > 0;$$

$$\frac{\partial \dot{\pi}^U}{\partial R} > 0, \frac{\partial \pi^U}{\partial R} > 0, \frac{\partial \ddot{\pi}^U}{\partial R} > 0, \frac{\partial \dot{\pi}^U}{\partial \lambda} > 0, \frac{\partial \pi^U}{\partial \lambda} > 0, \frac{\partial \ddot{\pi}^U}{\partial \lambda} > 0, \frac{\partial \dot{\pi}^U}{\partial M} > 0, \frac{\partial \pi^U}{\partial M} > 0, \frac{\partial \ddot{\pi}^U}{\partial M} > 0, \frac{\partial \dot{\pi}^U}{\partial \beta} > 0, \frac{\partial \pi^U}{\partial \beta} > 0, \frac{\partial \ddot{\pi}^U}{\partial \beta} > 0.$$

\[\blacksquare\]

B Proofs of the open-economy equilibrium

B.1 Proof of equations (10) and (11)

While the Bellman equations of upstream firms are the same as before, those for downstream firms are

$$r\hat{V}^D = \hat{\pi}^D + \lambda \left(V^D - \hat{V}^D\right) + \hat{V}^D,$$

$$rV^D = \pi + \mu^D \left(V^D - F_x - V^D\right) + V^D,$$

where $\hat{\pi}^D = \hat{\pi}^D + \hat{\pi}^D$. In steady state, $\dot{\hat{V}}^D = V^D = \dot{V}^U = 0$, and thus the value functions for downstream firms are given by

$$r\hat{V}^D = \left(r + \mu^D\right)\hat{\pi}^D + \frac{\lambda\pi}{r + \lambda + \mu^D} - \frac{\lambda\mu^D}{r + \lambda + \mu^D} F_x,$$

$$rV^D = \left(r + \mu^D\right)\pi + \frac{\mu^D \hat{\pi}^D}{r + \lambda + \mu^D} - \frac{\mu^D (r + \lambda)}{r + \lambda + \mu^D} F_x,$$

which can be written as

$$r(\hat{V}^D - F_x) = \pi + \frac{(r + \mu^D) \left[\hat{\pi}^D - \pi - (r + \lambda)F_x\right]}{r + \lambda + \mu^D},$$

$$rV^D = \pi + \frac{\mu^D \left[\hat{\pi}^D - \pi - (r + \lambda)F_x\right]}{r + \lambda + \mu^D}. \quad (B.1)$$

As in the closed economy, the profit sharing is uniquely determined by symmetric Nash bargaining:

$$(\hat{\pi}^D, \hat{\pi}^U) = \arg\max_{\pi^D,\pi^U} \left(\hat{V}^D' - F_x - V^D\right)\left(\hat{V}^U' - K - V^U\right).$$

subject to $\hat{\pi}^D' + \hat{\pi}^U' = \hat{\pi}$. Following the similar steps in Appendix A.1, this gives

$$\hat{\pi}^D' = \frac{1}{2} \left[\hat{\pi} + r \left(V^D - V^U\right) - (r + \lambda)(F_x + K)\right],$$

$$\hat{\pi}^U' = \frac{1}{2} \left[\hat{\pi} + r \left(V^U - V^D\right) + (r + \lambda)(F_x + K)\right].$$
Evaluating these at $\tilde{\pi}' = \tilde{\pi}$ and $\hat{\pi}' = \hat{\pi} = \tilde{\pi} - \tilde{\pi}'$,

$$\tilde{\pi}^D = \frac{1}{2(r + \lambda) + \mu^D + \mu^U} \times \left\{ (r + \lambda + \mu^D)\tilde{\pi} + (r + \lambda + \mu^U)\pi + (r + \lambda)[(r + \lambda + \mu^D)F_x - (r + \lambda + \mu^D)K] \right\},$$

$$\tilde{\pi}^U = \frac{1}{2(r + \lambda) + \mu^D + \mu^U} \times \left\{ (r + \lambda + \mu^U)\tilde{\pi} - (r + \lambda + \mu^U)\pi - (r + \lambda)[(r + \lambda + \mu^U)F_x - (r + \lambda + \mu^D)K] \right\}.$$ 

Then, rewrite these profit sharing rules as

$$\tilde{\pi}^D - \pi - (r + \lambda)F_x = \frac{(r + \lambda + \mu^D)[\tilde{\pi} - \pi - (r + \lambda)(F_x + K)]}{2(r + \lambda) + \mu^D + \mu^U},$$

$$\tilde{\pi}^U - (r + \lambda)K = \frac{(r + \lambda + \mu^U)[\tilde{\pi} - \pi - (r + \lambda)(F_x + K)]}{2(r + \lambda) + \mu^D + \mu^U}.$$

Substituting the above equalities into (B.1) gives

$$rV^D = \pi + \phi^D[\tilde{\pi} - \pi - (r + \lambda)(F_x + K)],$$

$$rV^U = \phi^U[\tilde{\pi} - \pi - (r + \lambda)(F_x + K)],$$

where the definition and property of $\phi^D$ and $\phi^U$ are exactly the same as before. Setting $rV^D = rF^D$ and $rV^U = rF^U$ and rearranging gives (10) and (11).

Next, we show that the above free entry condition holds even if the expected value of entry is used. From (B.1), it follows that

$$r(\tilde{V}^D - F_x) = \pi + \frac{(r + \mu^D)[\tilde{\pi} - \pi - (r + \lambda)(F_x + K)]}{2(r + \lambda) + \mu^D + \mu^U},$$

$$r(\tilde{V}^U - K) = \frac{(r + \mu^U)[\tilde{\pi} - \pi - (r + \lambda)(F_x + K)]}{2(r + \lambda) + \mu^D + \mu^U}.$$ 

Then, the expected values of entry are then given by

$$\mu^D r(\tilde{V}^D - F_x) + (1 - \mu^D)rV^D = (1 + r)\left\{ \pi + \phi^D[\tilde{\pi} - \pi - (r + \lambda)(F_x + K)] \right\}$$

$$= (1 + r)rV^D, $$

$$\mu^U r(\tilde{V}^U - K) + (1 - \mu^U)rV^U = (1 + r)\left\{ \phi^U[\tilde{\pi} - \pi - (r + \lambda)(F_x + K)] \right\}$$

$$= (1 + r)rV^U.$$ 

Setting $\mu^D r(\tilde{V}^D - F_x) + (1 - \mu^D)rV^D = rF^D$ and $\mu^U r(\tilde{V}^U - K) + (1 - \mu^U)rV^U = rF^U$, and solving for $\tilde{\pi}$ yields the equalities that are similar to (10) and (11) except for $\tilde{\phi}^D \equiv (1 + r)\phi^D$ and $\tilde{\phi}^U \equiv (1 + r)\phi^U$. \[ \square \]

### B.2 Proof of comparative statics

We first show that $\partial z/\partial r < 0$ and $\partial z/\partial F_x < 0$. Combining (10) with (11) and rearranging,

$$r(\phi^U F^D - \phi^D F^U)[\gamma(1 + \tau^{1-\sigma}) - 1] = rF^U + \phi^U(r + \lambda)(F_x + K).$$

Differentiating the above equality with respect to $\gamma$ and $F_x$ yields

$$\frac{\partial z}{\partial \gamma} = \frac{1}{(\sigma - 1)\tau^{1-\sigma}} \frac{\gamma r(\phi^U F^D - \phi^D F^U)}{\Omega}, \quad \frac{\partial z}{\partial F_x} = \frac{\phi^U(r + \lambda)}{\Omega},$$

where $\Omega \equiv \phi^U(z)\left\{ \gamma(1 + \tau^{1-\sigma}) - 1 \right\} rF^D - (r + \lambda)(F_x + K) - \phi^D(z)\left[ \gamma(1 + \tau^{1-\sigma}) - 1 \right] rF^U$, which is negative from $\mu^D(z) > 0$, $\mu^U(z) < 0$, $\phi^U F^D - \phi^D F^U > 0$ and $\left[ \gamma(1 + \tau^{1-\sigma}) - 1 \right] rF^D - (r + \lambda)(F_x + K) > 0$ (under condition (12)). The result directly follows from the above.
Next, we show that \( \partial^2 z / \partial \tau \partial \gamma > 0 \) and \( \partial^2 z / \partial F \partial \gamma > 0 \). Using \( \phi^{D'}(z) > 0 \) and \( \phi^{U'}(z) \) from the concavity of search technology \( \nu(\cdot, \cdot) \), we know that \( \partial \Omega / \partial \gamma < 0 \). Then, differentiating (B.2) with respect to \( \gamma \) gives

\[
\begin{align*}
\frac{\partial}{\partial \gamma} \left( \frac{\partial z}{\partial \tau} \right) &= \frac{1}{(\sigma - 1)\tau - \sigma} \frac{r}{\Omega^2} \left[ (\phi^U F^D - \phi^D F^U) \left( \Omega - \gamma \frac{\partial \Omega}{\partial \gamma} \right) + (\phi^{U'} F^D - \phi^{D'} F^U)(1 + \tau - \sigma) \frac{\partial \Omega}{\partial \gamma} \right], \\
\frac{\partial}{\partial \gamma} \left( \frac{\partial z}{\partial F} \right) &= \frac{(r + \lambda)}{\Omega^2} \left( \phi^{D'} \frac{\partial z}{\partial \gamma} - \phi^{D} \frac{\partial \Omega}{\partial \gamma} \right). 
\end{align*}
\]

The result follows from noting that \( \partial z / \partial \gamma > 0 \), \( \Omega < 0 \) and \( \Omega > \gamma \partial \Omega / \partial \gamma \).

Finally, we show that \( \partial^2 \pi / \partial \tau \partial \gamma < 0 \) and \( \partial^2 \pi / \partial F \partial \gamma < 0 \). To see these, it follows from (10') and (11') that

\[
\frac{\phi^D}{\phi^U} = z = \frac{r F^D - \pi}{r F^U} \iff \pi = r(F^D - z F^U).
\]

Differentiating this with respect to \( \tau \) and \( F \) yields

\[
\frac{\partial \pi}{\partial \tau} = -\frac{\partial z}{\partial \tau} F^U > 0, \quad \frac{\partial \pi}{\partial F} = -\frac{\partial z}{\partial F} F^U > 0.
\]

Furthermore, differentiating these with respect to \( \gamma \) gives

\[
\frac{\partial}{\partial \gamma} \left( \frac{\partial \pi}{\partial \tau} \right) = \frac{\partial}{\partial \gamma} \left( \frac{\partial z}{\partial \tau} \right) F^U, \quad \frac{\partial}{\partial \gamma} \left( \frac{\partial \pi}{\partial F} \right) = \frac{\partial}{\partial \gamma} \left( \frac{\partial z}{\partial F} \right) F^U.
\]

The result follows from noting that \( \partial^2 z / \partial \tau \partial \gamma > 0 \) and \( \partial^2 z / \partial F \partial \gamma > 0 \) as shown above. \( \square \)