Learning by Doing and Fragmentation

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Abstract

We analyze the competitive equilibrium and socially optimal allocation of production fragments in a two country model where there is learning by doing with spillovers between fragments in the home country. We distinguish between forward and backward knowledge linkages, where learning results from producing products that are less (more) complex than the current knowledge frontier with forward (backward) linkages. We compare the pattern of comparative advantage in the competitive and socially optimal cases, and compare the intensive and extensive margin of fragments produced at home in the two case. We establish a sufficient condition for the range of fragments produced at home to be non-decreasing with forward linkages. We also show that with backward linkages, the home country will produce some fragments in the neighborhood of the steady state that are more complex than those produced in the steady state.

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1 Introduction

Jones and Kierzkowski (1990) were the first to focus on the "fragmentation" of production, where the reduction of communication and transportation costs has made it possible to break up the production of a good into different fragments that are produced in different locations around the world. This phenomenon, which also has been referred to as vertical specialization and international outsourcing, has been widely studied in the theoretical literature. The focus in the literature on fragmentation has been primarily static, in that it examines the factors that are responsible for the existence of fragmentation and its implications for the pattern of trade and income distribution.\footnote{Jones and Kierzkowski (1990, 2001) and Golub, Jones and Kierzkowski (2007) emphasize the role of service links in connecting fragmented production blocks and scale economies. Deardorff (2001), Bond (2005), Grossman and Rossi-hansberg (2008) discuss the relationship between fragmentation and comparative advantage. Grossman and Helpman (2002), Antras and Helpman (2004), and Bond (2008) focus on the contracting issues that arise due to the potential for holdup problems when portions of the production process are outsourced. Grossman and Rossi-hansberg (2010), Dei (2010), and Nighenthi, Ma and Dei (2011) study the role of complementarities in the process of matching workers to tasks.}

Our focus in this paper is on the dynamics of the fragmentation process, and is motivated by the observation that the range of fragments that a country produces expands over time, particularly in the rapidly growing economies of East Asia. The dynamics of the export marketing process has been studied by Wortzel and Wortzel (1981), who developed a five stage model of export marketing based on interviews with executives from developing East Asian economies in industries such as electronics, footwear, and apparel. They describe a process by which firms expand from assembly operations that rely on foreign firms for distribution to selling their own products in foreign markets through their own distribution channels. At each stage, the firm internalizes some portions of the marketing and distribution process that had been performed by the foreign buyers or manufacturers in the previous stage. Hobday (1995) developed a similar model that focused on how late entering firms from developing economies acquired technology. At early stages, local firms engage in original equipment manufacture in which they produce products to the exact specifications of the foreign firm. Over time, these firms make modifications in the process and eventually advance to the stage of develop their own products that are sold under the foreign firm’s brand name. The most advanced stages involve development of their own brands.

The common feature underlying these stage models is the notion that firms in a developing country progress from performing the simplest fragments of the production process to increasingly complex fragments as they accumulate experience with the production process. One of our goals in this paper is to develop a trade model in which the range of fragments a country produces expands endogenously over time as a result of learning by doing in the production process, as suggested by the stage models. A second goal is to study the role of knowledge spillovers between firms, and to characterize optimal policy intervention in the face of knowledge spillovers. The explosive growth of the garment industry in Bangladesh described in Easterly (2001, Chapter 8) is a prime example of the importance of knowledge spillovers in
the dynamic process. The industry began with a joint venture between Daewoo and a local firm in 1980 to provide low wage production labor. In the following seven years the industry output grew by a factor of more than 50. Workers trained by Daewoo acquired knowledge about the production process and the financing of exports that allowed them to start their own firms and export directly. This example suggests that Daewoo captured only a portion of the benefits from the knowledge created by their entry and production. Therefore, it is also of interest to examine the optimal industrial policy in the presence of learning spillovers.

To address these questions, we employ a two country, two good Ricardian model in which the production in one sector requires the completion of a continuum of production tasks that can be ranked in terms of increasing complexity. The foreign country is assumed to have achieved its maximum productivity level in all tasks, but home country productivity improves over time due to the accumulation of knowledge through learning by doing. We make two key assumptions about the learning by doing process. The first is that there is a spillover of learning effects from simple fragments to more complex fragments, with productivity improvements from knowledge accumulation being exhausted sooner in the less complex fragments. This feature of the technology gives rise to the expansion of activities produced in the home country over time. The second is that the contribution of simple fragments to knowledge declines as the stock of knowledge increases. We show that the chain of comparative advantage when learning effects are external to the firm (the competitive equilibrium) will differ from the one obtained when learning effects are internalized (the socially optimal equilibrium) because the effective cost of labor is lower on fragments whose production contributes to knowledge.

We also show how the dynamics of the accumulation process differ depending on whether there are “forward knowledge linkages” or “backward knowledge linkages.” Forward knowledge linkages arise when a fragment continues to contribute to knowledge (and hence the productivity of more complex fragments) even after it has achieved its maximum productivity. The case of backward knowledge linkages occurs if achieving maximum productivity in a fragment requires production experience with more complex fragments. We show that when there are forward knowledge linkages, the steady state pattern of trade will be the same whether or not learning effects are internalized. However, the socially optimal path will involve temporary subsidies to fragments that contribute to knowledge, resulting in a more rapid accumulation of knowledge than in the competitive equilibrium. With backward knowledge linkages, in contrast, the steady state pattern of trade will involve a larger range of fragments when learning effects are internalized than when they are not. In particular, the transition to the steady state will involve the production of fragments that are more complex than those produced in the steady state when learning effects are internalized. Although the backward linkages result in the exit of some fragments once the subsidies are eliminated in the socially optimal policy, the production of these fragments on the transition path do not represent "mistakes" because of their contribution to the stock of knowledge.

Our analysis is related to other work on the dynamics of comparative advantage with learning by doing. Young (1991) analyzed the impact of trade on growth and welfare in the case where learning effects are not internalized, and shows that trade may be welfare reducing for a country if it loses the industries where learning effects are largest. Our specification
of the learning technology is similar to that introduced by Young, although he assumes that
the productivity improvements in a production activity are exhausted at exactly the same
point at which the activity ceases to contribute to knowledge. Thus, there is no possibility of
either forward or backward knowledge linkages. Our analysis also differs in that he assumes
that all goods are final goods. Goh and Wan (2005) point out that trade is more likely to be
beneficial when learning by doing arises in sectors where fragmentation is possible. Our work
differs in that they limit the model to two fragments, and do not consider the possibility of
forward and backward linkages or the role of internalization of learning effects. Ishikawa (1992)
and Bond, Jones and Wang (2005) analyze the effects of learning by doing on productivity in
service activities, which are an input to the exporting of labor intensive goods. They show that
the accumulation of knowledge results in a shift in production away from traditional export
activities that do not require service inputs over time, and toward more service intensive export
products.

The optimal commercial policy we obtain also bears some similarity to that obtained by
Housmann and Rodrik (2003), who consider a model where a country learns its productivity in
production of a good from the observation of the initial entrant. The competitive equilibrium
under-provides entry in their model, because entry provides information to other producers
about whether a particular activity is profitable. The competitive equilibrium provides too
much diversification in their model after entry due to an imitation delay that allows firms
to continue to produce even though they will ultimately be unprofitable. Our model also
provides insufficient entry along the transition path in the competitive equilibrium, although
the steady state effect depends on whether there are forward or backward linkages. We also
have the possibility of the exit of some activities after an initial period of production when
there are backward linkages, but exit in our model results from exhaustion of learning benefits
as opposed to learning of the true productivity level.

Section 2 introduces the model, and section 3 characterizes the competitive equilibrium
with free trade when learning effects are not internalized. Section 4 characterizes the chain
of comparative advantage and the socially optimal accumulation of knowledge when learning
effects are internalized. Section 5 offers some concluding remarks.

2 Model

We consider a two country, two good Ricardian model of international trade. We refer to the
two countries as home and foreign, and denote foreign variables with a “*”. The labor endow-
ments of the home and foreign countries are denoted by \( L \) and \( L^* \), respectively. Preferences
for the goods are assumed to take a quasi-linear form, \( U = C_1 + u(C_2) \) \( (U^* = C_1^* + u^*(C_2^*)) \),
with \( u \) (\( u^* \)) a strictly concave function. Good 1 is a standardized good that has only one
stage, and whose unit labor requirements are given by \( a_1 \) (\( a_1^* \)) in the home (foreign) country.
Sector 2 has a production technology that requires a continuum of “fragments” indexed by
\( s \in [0, 1] \), with each unit of good 2 requiring input of 1 unit of each of the fragments. The in-
dexing of fragments should be thought of as ordering fragments in increasing complexity, with
more complex fragments requiring greater skill on the part of labor. For example, the lowest
values of $s$ might correspond to simple assembly operations. Intermediate values could represent more sophisticated production activities, and the highest levels would involve engineering improvements and designs.

The foreign country is assumed to have exhausted all of its benefits from learning by doing in production of good 2, so its unit labor requirements, $b^*(s)$ will be unchanging over time. The home country unit labor requirements for the fragments of good 2, in contrast, will be changing over time as it gains experience in the production of good 2. We assume that the current state of production experience can be summarized by $K$, which can be interpreted as the stock of knowledge. The home unit labor requirements for segment $s$ will be expressed as $b(s,K)$, where $b_K(s,K) \leq 0$. We will work with the following specific functional form and parameter restrictions in the subsequent analysis:

**Assumption 1** The home and foreign technologies satisfy
(a) $b^*(s) = e^{\alpha^* s}$
(b) $b(s,K) = e^{\alpha s + \gamma \max(s-K,0)}$, where $\alpha - \alpha^* > 0$ and $\gamma > 0$

With these parameter restrictions, $\frac{b(s,K)}{b^*(s)}$ is increasing in $s$ for all $K$. Thus, the index $s$ ranks fragments in order of decreasing home country comparative advantage (according to the traditional Ricardian definition) for any given $K$. The assumption $\alpha - \alpha^* > 0$ implies an underlying relative advantage of foreign labor in the more complex fragments even after all of the gains from learning have been exhausted, as might arise if the foreign labor force is more educated. The magnitude of $\alpha^*$ determines the overall complexity of the requirements for good 2 as a whole, since the foreign labor requirement is increasing (decreasing) with the complexity of the fragment for $\alpha^* > 0$ ($\alpha^* < 0$).

We have normalized our index of knowledge, $K$, such that it represents the most complex fragment for which the home country has exhausted all of the gains from learning. We will refer to the fragment $s = K$ as the knowledge frontier. The accumulation of knowledge at home will raise output per worker at home for all fragments that are beyond the current frontier. This approach treats the stock of knowledge as a local public good within sector 2, which in a learning by doing framework means that knowledge accumulated from production in one fragment has spillovers to other production fragments. Our formulation ensures that the gains from knowledge are exhausted at some finite level of knowledge for each fragment, and that the exhaustion of gains from knowledge occurs at a higher level for more complex fragments. The impact of knowledge accumulation on relative home productivity in good 2 is illustrated in Figure 1. The $\frac{b(s,1)}{b^*(s)}$ locus shows the relative home labor cost in fragments if the home country achieves the maximal frontier of $K = 1$. At an initial stock of $K_1 < 1$, home has not achieved its minimal labor cost for $s > K_1$. As knowledge accumulates, relative home labor cost declines for all fragments beyond the current frontier.

The accumulation of knowledge is assumed to be the result of learning by doing,

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\footnote{With our normalization of $s$ to the unit interval, we have the possibility of $K < 0$ in cases where knowledge is sufficiently low that even the simplest fragments have not reached maximum productivity.}
\[ K(t) = \int_{0}^{1} \varphi(s, K) L_2(s, t) ds \]  

where \( L_2(s, t) \) is the amount of home country labor allocated to sector \( s \) at time \( t \) and \( \varphi(s, K) \geq 0 \) is a weight reflecting the contribution of fragment \( s \) employment to knowledge. We would not expect all fragments to contribute equally to the stock of knowledge. In particular, the amount of knowledge one gains from engaging in a simple task such as assembly should diminish as knowledge accumulates and more complex fragments are being performed. We capture this feature of learning by assuming that only the performance of fragments more complex than a threshold level will contribute to knowledge, and that this threshold increases as the stock of knowledge accumulates.

**Assumption 2** The knowledge production weights are given by

\[ \varphi(s, K) = \begin{cases} 
1 & s \geq K + \Delta \\
0 & s < K + \Delta 
\end{cases} \]  

We refer to \( \Delta > 0 \) as backward knowledge linkages and \( \Delta < 0 \) as forward knowledge linkages.

To see the importance of backward and forward knowledge linkages, suppose that the home country allocates an amount of labor \( L_2(s) > 0 \) for \( s \in [0, m] \) for all \( t \), with \( L_2(s) = 0 \) for \( s > m \). If \( \Delta < 0 \), the home country will eventually achieve the maximum productivity in all fragments that it produces. This occurs because some fragments that are inside the knowledge frontier continue to contribute to knowledge accumulation in the case of forward linkages. In contrast, with \( \Delta < 0 \) the fragments in the interval \([m - \Delta, m]\) will not achieve their maximum productivity because knowledge accumulation ceases once the knowledge stock reaches \( m - \Delta \). With backward knowledge linkages, fragments beyond the current knowledge frontier must be produced in order for knowledge to accumulate.

Our assumptions on knowledge accumulation and productivity improvement are similar to that in the learning by doing model of Young (1991), who examines spillovers between sectors that generates sustained growth. Our specification differs in that he only considers the case of \( \Delta = 0 \) in the learning technology, and he assumes that \( \alpha = \alpha^* \). Our specification can be contrasted with one in which learning by doing is fragment-specific, with \( K(s) \) denoting the stock of knowledge in sector \( s \) and \( \dot{K}(s) = L(s) \). If the home country initially produced the fragments \([0, s_1]\), its comparative advantage in these fragments would increase over time but would never expand beyond the initial range. Our specification can also be compared with that of Goh and Wan (2005), who consider spillovers from learning by doing in a model with two production fragments. Their formulation is equivalent to \( b(s, K) = h(s)g(K) \) and \( \varphi(s, K) = f(s) \) in the continuous case.\(^3\) This implies that knowledge is a pure public good that raise the productivity of all fragments proportionally for all levels of knowledge. If \( g(K) \) achieves its

\(^3\)Specifically, Goh and Wan (2005) consider the case with \( s \in \{1, 2\} \) and \( g(K) = e^{-K} \).
upper bound at a finite value $K_{\text{max}}$, all fragments will exhaust the benefits of learning at the same time. We opt for a framework in that allows knowledge to have differential impacts across the fragments. Goh and Wan capture the learning effects of different fragments by assuming that $f(s)$ is increasing in $s$, which implies that knowledge more rapidly by producing more complex fragments. The contribution of a fragment to the accumulation of knowledge will never be exhausted with their formulation, so that forward and backward knowledge linkages always exist at the same time.

3 The Competitive Equilibrium with Trade

We begin with the case of a competitive equilibrium with free trade, assuming that the benefits of learning are external to the firm. Under this assumption, the cost of labor to produce a unit of fragment $s$ is $wb(s, K)$ in the home country and $w^*b^*(s)$ in the foreign country, where $w$ and $w^*$ denote the respective wage rates. Similarly, the cost of good 1 will be $wa_1$ at home and $w^*a_1^*$ abroad. The pattern of specialization can be determined from Figure 1, because home will have comparative advantage in all activities for which the relative home labor cost is less than the relative foreign labor cost, $\frac{w}{w^*}$.

In order to simplify by reducing the number of cases to be considered, we impose the
following conditions on preferences and technologies:

**Assumption 3**: (Competitive Equilibrium Assumptions)

(a) both countries produce good 1 at a free trade equilibrium
(b) $e^{\gamma \max[-K_0, 0]} < \frac{a_1}{a_1} < e^{\alpha - \alpha^*}$, where $K_0$ is the initial stock of knowledge.

Assumption 3a will be satisfied if the demand for good 1 is sufficiently large relative to that for good 2, and implies that $w = \frac{1}{\tilde{a}}$ and $w^* = \frac{1}{\alpha^*}$ when good 1 is chosen as numeraire. Assumption 3b ensures that the initial stock of home country knowledge gives it comparative advantage in a proper subset of the fragments relative to good 1 at time 0, and that the foreign country maintains comparative advantage relative to good 1 in some fragments even when home has exhausted all learning gains for all fragments. The former inequality is required in order for the home country to ever be part of the production chain, because learning by doing is the only source of productivity gain in this model. The latter inequality, which means that the home country will never completely catch up in all fragments, is made to simplify notation by ruling out corner solution and can easily be relaxed.

The home country will produce all fragments for which $\int_{m(K, w)}^1 w b^*(s) ds$, which will be satisfied for all $s$ satisfying $e^{(\alpha - \alpha^*)s + \gamma (\max(s - K_0))} \leq \frac{w^*}{w}$. Assumption 3b guarantees that there will exist a marginal fragment $m \in (0, 1)$ such that this condition is satisfied for all $s \in [0, m]$. Defining $\bar{K}(w) = \frac{\log\left(\frac{w}{w^*}\right)}{\alpha - \alpha^*}$, we can solve for the marginal fragment produced:

$$m(K, w) = \begin{cases} 
\bar{K}(w) & K \geq \bar{K}(w) \\
\frac{\alpha - \alpha^*K + \gamma K}{\alpha + \gamma - \alpha^*} & K < \bar{K}(w)
\end{cases}$$  \hspace{1cm} (3)

The determination of the range of fragments produced at home is illustrated in Figure 1. At the knowledge stock $K_1 < \bar{K}(w)$, home will have comparative advantage for $s \in [0, m_1]$. For this level of knowledge, home produces an interval of fragments $[0, K_1]$ that have reached maximum productivity and an interval $(K_1, m_1)$ for which productivity can still be improved. An increment to knowledge at $K_1$ will be socially valuable, because it increases productivity in fragments in the interval $(K_1, m_1)$. At the knowledge stock $\bar{K}(w)$, the range of fragments produced coincides with the range of products that have achieved their maximum productivity level. Increments of knowledge will have no social value for $K > \bar{K}$, because further knowledge accumulation will not raise productivity in goods that are being produced.

The price of a unit of good 2 will be the sum of the costs of the individual fragments,

$$p(K) = \int_0^{m(K, w)} w b(s, K) ds + \int_{m(K, w)}^1 w^* b^*(s) ds$$  \hspace{1cm} (4)

Worldwide consumption of good 2 will be $C_2^W = D(p)$, where $D(p) = (u')^{-1} (p) + \left(\frac{u^*}{u}\right)^{-1} (p)$. Only the fragments in the interval $[K + \Delta, m(K, w)]$ will contribute to the accumulation of knowledge. The demand for labor to produce a unit of fragment $s$ is $D(p(K)) b(s, K)$, so we
can substitute into (1) to solve for the evolution of the stock of knowledge at time \( t \) in a competitive equilibrium

\[
\dot{K} = D(p(K)) \int_{K+\Delta}^{m(K,w)} b(s, K) ds
\]  

The stock of knowledge will continue to grow as long as \( K + \Delta < m(K, w) \). The steady state stock of knowledge will satisfy \( \dot{K} = 0 \), which occurs where \( m(K, w) = K + \Delta \). Utilizing (3), we obtain the steady state knowledge stock to be

\[
K^S(w) = \left\{ \begin{array}{ll}
\tilde{K}(w) - \frac{\Delta(\alpha + \gamma - \alpha^*)}{\alpha - \alpha^*} & \text{for } \Delta > 0 \\
\tilde{K}(w) - \Delta & \text{for } \Delta \leq 0
\end{array} \right.
\]  

The home country will produce all fragments in the interval \([0, m(K^S(w), w)]\) in the competitive steady state. The steady state stock of knowledge is decreasing in \( w \), because a higher cost of labor reduces the range of products that home can efficiently produce and reduces the return to knowledge accumulation.

\footnote{The price of good 2 will be decreasing as long as knowledge is accumulating from (4). Our assumption that good 1 is produced in both countries means that wage rates will be unaffected by the accumulated knowledge, so the gains from technical progress will be passed on to consumers in the free trade equilibrium.}
Figure 2 illustrates the evolution of the stock of knowledge in the competitive equilibrium for a case where $\Delta < 0$. The home country produces fragments in the interval $[0, m(K, w)]$, with the fragments in the interval $[0, K]$ having achieved the maximum efficiency. The stock of knowledge will continue to accumulate until it reaches the steady state level, $K^S(w)$, at the intersection of the $m(K, w)$ locus with the $K + \Delta$ line. In this case with forward knowledge linkages, all of the fragments produced in the steady state will have achieved maximum productivity because knowledge can be accumulated by producing goods that are beyond the current knowledge frontier. In particular, knowledge accumulated beyond $K$ is "useless" knowledge, in the sense that it does not contribute to lowering costs of fragments that are being produced.

Figure 3 shows the case of backward knowledge linkages, where $\Delta > 0$. In this case, knowledge can only be accumulated by producing fragments that are beyond the current knowledge frontier. Knowledge will accumulate until it reaches the steady state at $K^S(w)$, where the most complex fragment produced is $K - \frac{\gamma \Delta}{\alpha - \sigma^2}$. With backward linkages, the fragments $s \in [K, \tilde{K} - \frac{\gamma \Delta}{\alpha - \sigma^2}]$ do not achieve their maximum productivity in the steady state. Achieving maximum productivity in these fragments would require production of fragments $s' > m(K^S(w), w)$ in which home does not have static comparative advantage, and the myopic nature of the competitive equilibrium prevents the production of these fragments.

We can summarize the competitive equilibrium path as follows:

**Proposition 1** In a competitive equilibrium where learning is external to the firm and $K_0 < K^S(w)$
(a) Knowledge will accumulate until it reaches $K^S(w)$. The range of fragments produced at home will be non-decreasing over time, and will equal $[0, m(K^S(w), w)]$ in the steady state.

(b) If $\Delta < 0$, all fragments produced in the steady state will have achieved the minimum possible unit labor requirement. If $\Delta > 0$, the most complex fragments will not achieve the minimum unit labor requirement.

It should be noted that initial stocks $K > K^S(w)$ will also be competitive steady states. These cases are of less interest because the initial knowledge stocks are sufficiently high that the range of fragments produce result in no additional learning for the home country, and thus no dynamic to the fragmentation process.

4 The Social Planner’s Problem

We can contrast the allocation of labor in the competitive equilibrium with that which would be chosen by a social planner whose objective is to maximize world welfare. This solution will represent the outcome in a competitive equilibrium if the gains from learning are internal to the firm. If returns are not captured by the firm, then this outcome can be used to identify the subsidies required to attain the world welfare maximizing pattern of production and consumption.\footnote{An alternative formulation would be to analyze the welfare maximization problem for the home country. Since the home country has the ability to influence the cost of fragments and hence the world price of good 2, it has an incentive to use trade policies to manipulate the terms of trade as well as to capture learning spillovers. We analyze the social planners problem in order to focus on the role of learning spillovers.}

World welfare can be expressed as

$$U^W = \int_{0}^{\infty} \left[ C^W_1 + G(C^W_2) \right] e^{-\rho t} dt$$

(7)

where $G(C^W_2) = \max_{C^W_2} u(C^W_2 - C^*_2) + u^*(C^*_2)$. The planner will choose the time paths for employment to maximize (7) subject to the full employment constraints,

$$L \geq L_1(t) + \int_{0}^{1} L_2(s, t) ds$$

(8)

$$L^* \geq L_1^*(t) + \int_{0}^{1} L_2^*(s, t) ds$$

the technological constraints

$$C^W_1(t) \leq \frac{L_1(t)}{a} + \frac{L_1^*(t)}{a^*}$$

(9)

$$C^W_2(t) \leq \frac{L_2(s, t)}{b(s, K(t))} + \frac{L_2^*(s)}{b^*(s)}$$

and the knowledge accumulation condition (1).
Letting \( \mu \) denote the costate variable associated with the stock of knowledge, the solution to the social planner’s problem yields the condition that the home country will produce all fragments for which (see Appendix for details)

\[
(w - \mu(t) \varphi(s, K(t))) b(s, K(t)) \leq w^* b^*(s)
\]

The costate variable \( \mu \) has the interpretation of being the value of an increment of knowledge to world welfare. Condition (10) shows that the cost of employing home labor in the production of \( s \) is equal to the wage rate less the value of the knowledge created by employment in \( s \). For \( s < K + \Delta \), it is optimal for the home country to produce if \( b(s, K(t)) \geq w^* \). When \( \mu > 0 \), there is a discontinuity in the relative cost of home production at the boundary of activities that generate learning benefits, \( K + \Delta \), and the ranking of fragments by relative home labor cost is not monotone in \( s \). This leads to the possibility that the range of fragments in which home has comparative advantage is not a connected set, as established in the following result.

**Lemma 1:** The set of fragments \( \Omega(K, \mu) \), produced at home in the socially optimal production pattern, is

(a) \( s \in [0, m(K, w - \mu)] \) for \( K \leq K^S(w) \)

(b) \( s \in [0, m(K, w)] \) and \( s \in [K + \Delta, m(K, w - \mu)] \) for \( K \in (K^S(w), K^S(w - \mu)] \)

(c) \( s \in [0, m(K, w)] \) for \( K > K^S(w - \mu) \)

The proof of Lemma 1 can be illustrated with the help of Figure 4, which shows how the chain of comparative advantage in the dynamic case differs from that in the static one. The locus \((b(s, K(t)) - w^*) / b^*(s)\) shows the relative productivity of home labor in fragment \( s \) when the value of learning is taken into account. The home country has comparative advantage in all fragments for which this relative productivity is less than \( w^* / w \).

There are three possible types of specialization patterns that can arise in the dynamic case. If the relative foreign wage is sufficiently low that \( w^* / w \leq (w^*)_0 \), the home country will only produce fragments with \( s < K + \Delta \) for which no knowledge accumulation is taking place. This specialization pattern occurs when \( m(K, w - \mu) < K + \Delta \), which requires that \( e^{(\alpha - \alpha^*)(K + \Delta) + \gamma \max(\Delta, 0)} > \frac{w^*}{w - \mu} \). Using the definition in (6), this condition will be satisfied if \( K > K^S(w - \mu) \), which establishes part (c) of the Lemma. This case yields the same range of goods being produced as in the competitive case, and arises when the value of additional knowledge is sufficiently low that it is not worthwhile producing fragments in which there is static comparative disadvantage to obtain new knowledge. A second type of specialization pattern occurs if \( \frac{w^*}{w} > (\frac{w^*}{w})_1 \), where the home country produces a single interval of activities and obtains gains from learning for those with \( s > K + \Delta \). This specialization pattern occurs when \( m(K, w - \mu) > K + \Delta \), which requires that \( e^{(\alpha - \alpha^*)(K + \Delta) + \gamma \max(\Delta, 0)} < \frac{w^*}{w} \). This case
arises if $K \leq K^S(w)$, which yields part (a) of the Lemma. The final case arises where $(\frac{w^*}{w})_0 < \frac{w^*}{w} < (\frac{w^*}{w})_1$, with the home country producing two disjoint sets of fragments. This is the intermediate case in which knowledge is sufficiently valuable that home produces some goods in which it has static comparative disadvantage, but home productivity is not so high that it produces some fragments that have no learning benefits. Since $K^S(w)$ is decreasing in $w$, the three intervals of knowledge stocks identified in the Lemma will be non-empty for $\mu > 0$.

The socially optimal level of consumption will satisfy $C^W_2 = D(P)$, where

\[
P(K, \mu) = \int_0^1 \min[(w - \mu \varphi(s, K))b(s, K), w^*b^*(s))]ds
\]

where

\[
\frac{\partial P}{\partial \mu} = -\int_{K+\Delta}^{m(K,w-\mu)} b(s, K)ds
\]

\[
\frac{\partial P}{\partial K} = \int_{\Omega} (w - \mu \varphi(s, K)) b_K(s, K)ds
\]

For given $K$, the price on the socially optimal path will be lower than that on the competitive path if $\mu > 0$ and $m(K, w - \mu) > K + \Delta$. Increases in the stock of knowledge reduce the price of good 2 on the optimal path if the home country produces fragments that are outside the current knowledge frontier. For a given $K$, the socially optimal allocation will imply a greater range of fragments produced in the home country and a greater world consumption of good 2 than would arise in the competitive equilibrium if $\mu > 0$ and $K < K^S(w - \mu)$.

Implementation of the socially optimal production pattern can be achieved with a production subsidy of $\sigma(s) = \mu \varphi(s, K)b(s, K)$.

Evaluating (1) at the optimal production plan yields the evolution of the stock of knowledge
capital in the solution to the planner’s problem,

$$\dot{K} = \begin{cases} 
C_2^W \int_{K+\Delta}^{m(K,w-\mu)} b(s,K) \, ds & K < K^S(w-\mu) \\
0 & K \geq K^S(w-\mu) 
\end{cases}$$

(14)

Knowledge accumulation will be positive as long as the home country specializes in some fragments \(s > K + \Delta\), which from Lemma 1 requires \(K < K^S(w-\mu)\).

The evolution of the costate variable is given by

$$\dot{\mu} = \rho \mu - \frac{\partial H}{\partial K}$$

(15)

Equation (15) characterizes the change in the value of an increment of knowledge along the optimal path. This has the usual interpretation of being the difference between the cost of knowledge (\(\rho \mu\)) and the marginal product of knowledge (\(\frac{\partial H}{\partial K}\)). It is shown in the Appendix that for \(K < K^S(w)\),

$$\frac{\partial H}{\partial K} = -C_2^W \left( \frac{\partial P}{\partial K} + \mu b(K+\Delta,K) \right)$$

(16)

An increment of knowledge will raise the productivity of all domestic fragments for which \(s > K\), so the first term in (16) is the amount of cost reduction resulting from these productivity improvements. For \(m(K, w-\mu)\) an increment of knowledge will also eliminate the contribution of the fragment \(K + \Delta\) to knowledge if it is being produced. The marginal product of knowledge is the sum of these two effects.

### 4.1 Socially Optimal Steady States

A steady state of the social planner’s problem will exist if there is a pair \(\{\mu^{SO}, K^{SO}\}\) such that (14) and (15) are satisfied with \(\dot{\mu} = \dot{K} = 0\). It can be shown that \(\mu^{SO} = 0\). We will restrict our discussion to steady states for which \(K \leq \max[K(w), K^S(w)]\). Steady states with \(K > \max[K(w), K^S(w)]\) can only be reached with initial endowments of knowledge sufficiently large that there is no knowledge accumulation, and are thus of less interest.\(^6\)

The following result, which is proven in the Appendix, establishes the relationship between the socially optimal steady states and those in the competitive equilibrium.

**Proposition 2** (a) If \(\Delta \leq 0\), then the socially optimal steady state is \(\mu^{SO} = 0\) and \(K^{SS} = K^S(w)\), which is the same knowledge stock as in the competitive steady state.

(b) If \(\Delta > 0\), \(K > K^S(w)\) in any socially optimal steady state. There exists a \(K^{SO} \in (K^S(w), K)\), satisfying the conditions for a socially optimal steady state. For any such steady state, \(\mu^{SO} > 0\).

\(^6\)It is shown in the Appendix that all \(K > \max[K(w), K^S(w)]\) satisfy the conditions for a steady state equilibrium with \(\mu = 0\). These steady states also represent steady states in the competitive equilibrium, but can only be reached with \(K_o = K^{SS}\).
Proposition 1 established that if $\Delta \leq 0$, all fragments produced by the home country would achieve their maximum productivity in the steady state. Proposition 2a shows that the socially optimal steady states coincide with the competitive steady states when $\Delta \leq 0$, both in terms of the range of fragments produced and the output per fragment, because increments of knowledge have no social value in the competitive steady state. The myopic nature of the competitive equilibrium does not affect the steady state production pattern with forward knowledge linkages because the home country can accumulate the required knowledge by producing fragments in which it has comparative advantage in the steady state.

In contrast, the case of backward linkages ($\Delta > 0$) requires that goods more sophisticated than the marginal good be used to achieve the minimum cost for the marginal good (i.e. $m(K^S(w), w) > K^S(w)$) as illustrated in Figure 3). Since knowledge has positive value at $K^S(w)$, the socially optimal steady state will have a higher knowledge stock because the planner will choose production of more complex fragments in order to take advantage of the backward linkages they provide. Accumulating additional knowledge is costly in the steady state with $\Delta > 0$ because it requires production of fragments in which the home country does not have long run comparative advantage. As a result, the marginal fragment produced will not achieve its maximum productivity in steady states with $\mu^S > 0$. Steady states with $\mu^S > 0$ equate the benefit of additional knowledge accumulation with the cost of acquiring that knowledge.

Due to the potential for increasing returns to knowledge production, there may be more than one value of the knowledge stock satisfying $\mu^{SO} > 0$ and $K^{SO} = K^S(w - \mu^{SO})$. It is shown in the Appendix that this possibility arises if the right hand side of (24) is increasing in $K$ for some portion of the interval $[K^S(w), \bar{K}]$. There are two potential sources of increasing returns. One is that additional knowledge raises the output of good 2, which raises the increase of additional knowledge creation. The other is due to the possibility that more complex fragments represent a greater share in total production costs as $s$ increases, which can raise the return to knowledge creation.

4.2 Optimal Knowledge Accumulation

We can also compare the socially optimal and competitive paths on the transition to the steady state. We first consider the case with $\Delta \leq 0$ and an initial knowledge stock $K_0 < K^S(w) + \Delta$. At this initial knowledge stock, the marginal fragment that would be produced in the competitive equilibrium has not achieved its maximum productivity level because $m(K, w) > K$). The following result is proven in the Appendix.

**Proposition 3** Assume $\Delta \leq 0$ and $K_0 < K^S(w) + \Delta$,

(a) The socially optimal production plan will have $\mu^0 > 0$, and will expand both the extensive and intensive margin of production of good 2 in the home country relative to the competitive equilibrium.

(b) If the planner’s optimization problem is concave in $K$, the range of fragments produced is non-decreasing over time.
Figure 5: Optimal Path for $\Delta < 0$

Figure 5 illustrates the comparison between the competitive path and the socially optimal path in the case where $\Delta \leq 0$, given an initial value of the capital stock $K_0 < K^S(w) + \Delta$. Knowledge has positive value at the initial capital stock ($\mu_0 > 0$) because $m(K_0, w) > K_0$. The home country produces fragments in the interval $[0, m(K_0, w - \mu_0)]$, which exceeds the range produced in the competitive equilibrium at time 0. The cost of producing fragments in the interval $[K_0 + \Delta, m(K_0, w - \mu_0)]$ is evaluated using the shadow wage $w - \mu_0$ by the social planner, so the cost of good 2 given by (11) will be lower than in the competitive case as given by (4). The optimal production pattern could be obtained by offering a subsidy of amount $\mu_0$ to each unit of labor employed in fragments $s > K_0 + \Delta$, which would expand both the range of fragments produced and the output of each fragment in this interval relative to the competitive equilibrium at time 0. For the case where $\dot{H}(K, \mu)$ is concave in $K$, the value of $\mu$ will decline over time and the pattern of home production will follow a path as illustrated by the arrows in Figure 5. The socially optimal plan uses subsidies to expand both the intensive and extensive margins of production of fragments relative to the competitive equilibrium along the transition path, but these subsidies would be phased out as the steady state is approached.

In the case of backward knowledge linkages, the socially optimal path will expand the intensive and extensive margins relative to the competitive equilibrium both in the steady state (as noted in Proposition 2) and on the path to the steady state. However, the range of fragments produced will not be monotonic on the path to the steady state.

**Proposition 4** In a socially optimal equilibrium with $\Delta > 0$ and $K_0 < K^S(w)$:
(a) The socially optimal production plan will have $\mu^0 > 0$, and will expand both the extensive and intensive margin of production of good 2 in the home country relative to the competitive equilibrium.

(b) In the neighborhood of the steady state, the set of fragments produced by the home country will not be connected and will include some fragments that are more complex than in the steady state.

The argument for establishing part (b) of Proposition 4 is essentially the same as in the case of forward knowledge linkages. The fact that the shadow wage is less than the market wage on the path to the steady state results in a larger range of fragments produced and a larger output of each fragment. The difference between the two cases arises in the neighborhood of the steady state, as illustrated in Figure 6. At the steady state illustrated by $K^S(w - \mu^{SO})$, knowledge has positive value and the home country produces fragments $s \in [0, m(K^S(w - \mu^{SO}), w)]$. The steady state knowledge stock exceeds that in the competitive equilibrium, but the maximum productivity is not achieved for $s \in [K, m(K^S(w - \mu^{SO}), w)]$. At the steady state, the value of an increment of knowledge is equated to the cost of producing that knowledge. The cost of producing knowledge with backward linkages is the production of fragments whose static cost exceeds that in the foreign country. Note that for $K \in (K^S(w), K^S(w - \mu^{SO}))$, the optimal production pattern will involve producing a range of fragments $[0, m(K, w)]$ for which no learning takes place and a disjoint set of fragments $[K + \Delta, m(K, w - \mu)]$ that contribute to
knowledge accumulation for any $\mu > 0$. The specialization pattern in this range is of the type identified in Lemma 1(b). fragments in the interval $n$ contrast, the most complex fragment that is being produced in the competitive steady state will not have achieved its maximum efficiency because $m(K, w) > K$. For example, $m(K^S(w), w) > K^S(w)$ as illustrated in Figure 6.

5 Conclusions

Our analysis has shown how knowledge spillovers between fragments of differing complexity in the production chain can generate the expansion of the extensive and intensive margins of production over time as knowledge is accumulated from production. One point of emphasis in the analysis has been on how the static pattern of comparative advantage, which ignores the benefits of knowledge accumulation on the cost of production, differs from the dynamic comparative advantage. With the technology specification we examine, the set of industries in which the home country has comparative advantage is a connected set under the static definition of comparative advantage. However, the pattern of comparative advantage under the dynamic definition depends on both the current productivity and the potential for contributing to knowledge, which may result in two disjoint sets of fragments.

A second point of concentration is how the direction of knowledge spillovers, whether backward or forward, affects the dynamics of the learning process and the socially optimal pattern of subsidies. In the case of forward knowledge linkages, socially optimal level of knowledge can be accumulated by specializing in activities in which the home country has static comparative advantage. However, the path to the socially optimal steady state can be improved by providing subsidies to activities that generate knowledge spillovers, with the subsidies being phased out as the steady state is approached. With backward knowledge linkages, achieving the socially optimal level of knowledge requires production in activities in which the home country has static comparative disadvantage. As a result, the competitive equilibrium will not achieve the socially optimal level of knowledge. The socially optimal path will result in an expansion of both the extensive and intensive margins of production relative to the competitive equilibrium both on the path to the steady state and in the steady itself. This can also be achieved by subsidies to knowledge creating activities that are phased out as the steady state is approached. An interesting feature of the backward linkage case is that the optimal plan will involve temporary production of fragments in which the country does not have long run comparative advantage.
6 Appendix

Derivation of necessary conditions for the Planner’s Problem and Proof of Lemma 1: The constant marginal utility of good 1 ensures that (9) and (8) hold with equality in any solution, since consumers will never be satiated.

The following result simplifies the problem by showing that an optimal allocation must result in production of at most two intervals of fragments by the home country.

Lemma 1 If it is optimal for home to produce \( s_0 < K + \Delta \), then it is also optimal to produce all \( s \in [0, s_0] \). If it is optimal for home to produce \( s_1 > K + \Delta \), then it is optimal for home to produce \( s \in [K + \Delta, s_1] \).

Proof. Suppose there is an allocation of labor such that \( L_2(s') = 0 \) for \( s' < s_0 \), \( L_2(s_0) = 0 \), and (9) is satisfied with equality at \( s' \) and \( s_0 \). A reduction in the amount of home labor employed in fragment \( s_0 \) of \( dx \) and a corresponding increase in foreign labor of amount \( \frac{b^*(s_0)}{b^*(s_0, K)} dx \) will keep output of fragment \( s_0 \) constant. Applying the home labor to production of fragment \( s' \) results in an increase in output of fragment \( s' \) of \( \frac{dx}{b^*(s', K)} \) and allows a release of foreign labor of amount \( \frac{b^*(s')dx}{b^*(s', K)} \). Since \( \frac{b^*(s_0)}{b^*(s_0, K)} < \frac{b^*(s')dx}{b^*(s', K)} \), the released foreign labor is sufficient to hold output of fragment \( s_0 \) constant and also increase output of good 1. This reallocation has no effect on knowledge accumulation, so the initial allocation could not have been optimal. Since labor employed in fragments \( s > K + \Delta \), all have the same effect on knowledge accumulation, a similar argument can be made for these fragments. 

Let \( s_N \in [0, K + \Delta] \) denote the most complex fragment that home produces that does not contribute to learning, and \( s_L \in [K + \Delta, 1] \) the most complex fragment produced that contributes to learning. We can then define the set of home fragments to be \( \Omega(s_L, s_N) = [0, s_N] \cup [K + \Delta, s_L] \) and the set of foreign fragments to be \( \Omega^*(s_L, s_N) = [s_N, K + \Delta] \cup [s_L, 1] \). Since (8) will hold with equality, we have \( L_2(s) = b(s, K)C_2^W \) for \( s \in \Omega(s_L, s_N) \). We can then express the Hamiltonian for this problem as

\[
H(s_L, s_N, C_2^W, K, \mu) = G(C_2^W) + \frac{L_1}{a_1} + \frac{L_1^*}{a_1^*} + \mu C_2^W \int_{K+\Delta}^{s_L} b(s, K) ds
+ \psi_N(K + \Delta - s_N) + \psi_L(s_L - K - \Delta) + w \left( L - L_1 - C_2^W \int_{\Omega(s_L, s_N)} b(s, K) ds \right)
+ w^* \left( L^* - L_1^* - C_2^W \int_{\Omega^*(s_L, s_N)} b^*(s) ds \right)
\]

Note that the constraints \( s_N \leq K + \Delta \) and \( s_L \geq K + \Delta \) have been added to the problem, with the corresponding multipliers \( \psi_N \) and \( \psi_L \). The constraints \( s_N \geq 0 \) and \( s_L \leq 1 \) are assumed to be satisfied. In this derivation, we will assume an interior solution with \( L_1, L_1^* > 0 \), so that the multipliers on the the full employment constraints will satisfy \( w = \frac{1}{a_1} \) and \( w^* = \frac{1}{a_1^*} \).

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The necessary conditions for the choice of the remaining control variables for this problem are

\[(i) \frac{\partial H}{\partial s_N} = C_2^W (w^* b^*(s_N) - wb(s_N, K)) - \psi_N = 0, \quad \psi_N (K + \Delta - s_N) = 0 \quad (18)\]

\[(ii) \frac{\partial H}{\partial s_L} = C_2^W (w^* b^*(s_L) - (w - \mu)b(s_L, K)) + \psi_L = 0, \quad \psi_L (s_L - K - \Delta) = 0 \quad (19)\]

\[(iii) \frac{\partial H}{\partial C_2^W} = G'(C_2^W) - \left( \int_{s_N}^{s_L} wb(s, K) ds + \int_{K+\Delta}^{s_L} (w - \mu)b(s, K) ds \right) \]

\[\quad - \left( \int_{s_N}^{K+\Delta} w^* b^*(s) ds + \int_{s_L}^{1} w^* b^*(s) ds \right) = 0 \quad (20)\]

Condition (i) indicates that fragments without learning benefits at home will be produced by the country with the lower labor cost. Using the definitions of \(m(K, w)\) and \(K^S(w)\), this solution can be expressed as

\[s_N(K) = \begin{cases} 
  m(K, w) & K > K^S(w) \\
  K + \Delta & K \leq K^S(w)
\end{cases} \quad (19)\]

Condition (ii) indicates that fragments that contribute to knowledge will be produced at home if the home labor cost is lower when evaluated at the shadow wage \(w - \mu\), which gives

\[s_L(K, \mu) = \begin{cases} 
  K + \Delta & K > K^S(w - \mu) \\
  m(K, w - \mu) & K \leq K^S(w - \mu)
\end{cases} \quad (20)\]

Combining (19) and (20) yields the solution for \(s\) in Lemma 1.

Condition (iii) can be expressed as \(G'(C_2^W) = \int_0^1 \min[(w - \mu\varphi(s, K))]b(s, K), w^* b^*(s)] ds,\) which yields (11). Since \(G\) is strictly concave, (iii) can be inverted to obtain \(C_2^W(P)\) as in the text.

The evolution of the costate variable is given by \(\dot{\mu} = \rho\mu - \frac{\partial H}{\partial K},\) where

\[\frac{\partial H}{\partial K} = -C_2 \int_{\Omega(s_L, s_N)} (w - \mu\varphi(s, K)) b_K(s, K) ds \]

\[+ C_2 ((w - \mu)b(K + \Delta, K) - w^* b^*(K + \Delta)) + \psi_N - \psi_L \quad (21)\]

The first term in (21) is the amount of cost reduction that results from an increment of knowledge. The remaining terms represent the impact of an increment of knowledge on the cost of the marginal good for which learning occurs, \(K + \Delta\). The value of the latter term will vary depending on the pattern of specialization. Using the fact that \(\psi_N = C_2^W (w^* b^*(s_N) - wb(s_N, K))\) for \(K < K^S(w)\) and \(\psi_L = C_2^W ((w - \mu)b(s_L, K) - w^* b^*(s_L))\) for \(K > K^S(w - \mu)\), (21) becomes
\[
\frac{\partial H}{\partial K} = \begin{cases} 
-C_2^W \left( \frac{\partial P}{\partial K} + \mu b(K + \Delta, K) \right) & K < K^S(w) \\
-C_2^W \left( \frac{\partial P}{\partial K} - (w - \mu) b(K + \Delta, K) + w^* b^*(K + \Delta) \right) & K \in (K^S(w), K^S(w - \mu)) \\
-C_2^W \frac{\partial P}{\partial K} & K > K^S(w - \mu)
\end{cases}
\]

Note that \( \frac{\partial H}{\partial K} \) will be continuous in \( K \) at \( K^S(w) \) because \( wb(K^S(w)+\Delta, K^S(w)) = w^* b^*(K^S(w)+\Delta) \). Similarly, \( \frac{\partial H}{\partial K} \) will be continuous in \( K \) at \( K^S(w - \mu) \) because \((w - \mu)b(K^S(w - \mu) + \Delta, K^S(w - \mu)) = w^* b^*(K^S(w - \mu) + \Delta) \).

**Sufficiency:**

Substituting the solutions for the controls into the Hamiltonian yields

\[
\tilde{H}(K, \mu) = H(s_N(K), s_L(K, \mu), C_2^W(K, \mu), K, \mu)
\]

Seierstad and Sydsaeter (1977) derive sufficient conditions for optimality which include the requirement that \( \tilde{H}(K, \mu) \) be concave in \( K \). Differentiation of (23) gives \( \frac{\partial \tilde{H}(K, \mu)}{\partial K} = \frac{\partial H(s_N(K), s_L(K, \mu), C_2^W(K, \mu), K, \mu)}{\partial K} \).

Consider the case of \( K \leq K^S(w) \), where from (22) we have

\[
\frac{\partial H}{\partial K} = -C_2(K, \mu) \left( \int_K^{K+\Delta} wb_K(s, K) ds + \int_{K+\Delta}^{s_L(K, \mu)} (w - \mu) b_K(s, K) ds + \mu b(K + \Delta, K) \right)
\]

Differentiating this expression

\[
\frac{\partial^2 H}{\partial K^2} = -\left( \frac{\partial C_2 \partial H}{\partial K \partial K} \right) - C_2(K, \mu) \left( \int_K^{K+\Delta} wb_K(K, K) ds + \int_{K+\Delta}^{s_L(K, \mu)} (w - \mu) b_K(s, K) ds \right)
\]

\[
-\frac{s_L(N, K, \mu)}{K} \left( wb_K(K, K) - (w - \mu) b_K(s_L(K, \mu), K) \frac{\partial s_L}{\partial K} \right)
\]

The first expression is the effect of increases in the stock of knowledge on demand for good 2, which will be positive if demand is not perfectly inelastic. This provides an element of increasing returns to the model. The second term is the effect of an increase in the stock of knowledge on the degree of cost reduction, which provides decreasing returns to knowledge accumulation with \( b_{KK}(s, K) > 0 \). The final term arises from the effect of an increment of knowledge on the range of product produced and can be either positive or negative.

**Proof of Proposition 2:**

There will be no knowledge accumulation if \( m(K, w - \mu) \leq K + \Delta \), which requires that \( K \geq K^S(w - \mu) \). Inverting this, we obtain the requirement that \( \mu \leq f(K) \equiv w - K^{-1}(K) \) for \( K \geq K^S(w) \), where \( f(K^S(w)) = 0 \) and \( f'(K) = (\alpha - \alpha^*) (w - \mu) > 0 \). From (22), \( \mu = 0 \) for \( K \geq K^S(w - \mu) \) requires that

\[
\mu = \frac{-w C_2(P(K, \mu)) \left( \int_K^{\max[m(K, w - \mu), K]} b_K(s, K) ds \right)}{\rho}, \quad (24)
\]
Since \( m(K, w) = \bar{K}(w) \) for \( K \geq \bar{K}(w) \), (24) will be satisfied with \( \mu = 0 \) for all \( K \geq \bar{K}(w) \). If \( \Delta \leq 0 \), then \( K^S(w) \geq \bar{K}(w) \) and the conditions for a steady state with \( \mu^{SO} = 0 \) are satisfied for all \( K \geq K^S(w) \). These steady states coincide with those in the competitive equilibrium. To show that there can be no steady states with \( \mu^{SO} > 0 \) in this case, note that \( m(K, w - \mu) = \bar{K}(w - \mu) \) for \( K \geq \bar{K}(w - \mu) \). Therefore, \( K \geq K^S(w - \mu) \) will not satisfy the conditions for a steady state with \( \mu > 0 \) because the right hand side of (24) will evaluate to \( 0 \). This establishes \( \text{(a)} \) of the Proposition.

If \( \Delta > 0 \), \( K^S(w) < \bar{K}(w) \) and \( m(K^S(w), w) > K^S(w) \). The competitive steady state will not satisfy the conditions for a socially optimal steady state because the right hand side of (24) is positive at \( K^S(w) \). Since \( K^S(w) < \bar{K}(w) \) in this case, the conditions for a socially optimal steady state are satisfied for \( K \geq \bar{K}(w) \) with \( \mu^{SO} = 0 \). To determine whether there are steady states on the interval \((K^S(w), \bar{K}) \) with \( \mu > 0 \), note that the set of fragments produced will fall in the region indicated by part 2 of Lemma 1, with \( s_L = m(K, w) \) at a steady state. The home country produces no fragments that are generating new knowledge in the steady state, so \( \frac{\partial P}{\partial \mu} = 0 \) and (24) can be expressed as

\[
\mu = g(K) = \begin{cases} 
\frac{wC_2(P(K,0))e^{\alpha K}}{\alpha + \gamma} (e^{(\alpha + \gamma)(m(K, w)-K)} - 1) > 0 & K \in [K^S(w), \bar{K}) \\
0 & K \geq \bar{K}
\end{cases}
\]  

(25)

The conditions for a steady state will be satisfied for any \( K \) such that \( f(K) \geq g(K) \) for \( K \in [K^S(w), \bar{K}(w)] \). The above discussion has established that \( g(K^S(w)) - f(K^S(w)) > 0 \) and \( g(\bar{K}(w)) - f(\bar{K}(w)) < 0 \), so a steady state exists with \( \mu = f(K) = g(K) > 0 \) by the continuity of \( f(.) \) and \( g(.) \). A sufficient condition for the solution to be unique is \( g'(K) \leq 0 \). If demand is perfectly inelastic, \( g'(K) < 0 \) requires that

\[
g'(K) = \frac{wC_2e^{\alpha K}}{\alpha + \gamma} \left[ \frac{\alpha (e^{(\alpha + \gamma)(m(K, w)-K)} - 1)}{\alpha + \gamma - \alpha^*} - \frac{(\alpha + \gamma) (\alpha - \alpha^*) e^{(\alpha + \gamma)(m(K, w)-K)}}{\alpha + \gamma - \alpha^*} \right] < 0
\]  

(26)

This condition must be satisfied in the neighborhood of \( \bar{K} \), because the term in brackets goes to \( -\frac{(\alpha + \gamma)(\alpha - \alpha^*)}{\alpha + \gamma - \alpha^*} < 0 \) at \( \bar{K} \). The bracketed expression is increasing in \( m(K, w) - K \), which is decreasing in \( K \), so the sufficient condition requires that (26) be satisfied at \( K^S(w) \):

\[
\Delta < \frac{\log \left( \frac{\alpha (\alpha + \gamma - \alpha^*)}{\alpha + \gamma} \right)}{\alpha + \gamma}
\]

If this condition is satisfied, there will be a unique value \( K^{SO} \) satisfying \( f(K^{SO}) = g(K^{SO}) \), and \( K \in [K^{SO}, \bar{K}) \) will be steady states with \( \mu = g(K^{SO}) \). Note by the way that if we consider the case with \( \alpha > \alpha^* = 0 \), (26) will always be satisfied.
For the case where demand is not perfectly inelastic, the condition will be more stringent. Specifically, it requires

\[
g'(K) = \frac{wC_2 e^{\alpha K}}{\alpha + \gamma} \left[ \left( \frac{C_1(P_2)}{C(P_2)} \frac{\partial P_2}{\partial K} + \alpha \right) \left( e^{(\alpha+\gamma)(m(K,w)-K) - 1} \right) \right] < 0 \tag{27}
\]

As in the previous case, this must be satisfied in the neighborhood of \( K = \bar{K} \).

The considerations here to establish uniqueness of the steady state are very similar to those required for sufficiency, since they involve the requirement that the Hamiltonian be concave in \( K \).

**Proof of Proposition 3:**

(a) Since \( m(K, w - \mu) \) is increasing in \( \mu \), the socially optimal range of fragments will exceed that for the competitive case for any given \( K \) when \( \mu > 0 \). For \( K < K^S(w) + \Delta \), we have \( \mu^S > 0 \) in the socially optimal solution.

(b) Differentiation of (3) for \( K < K^S(w) \) yields \( \dot{m} = \frac{\gamma \bar{K} + \mu}{\alpha - \alpha^* + \gamma} \). Evaluating this expression using (1) and (22) with \( \Omega(K,0) = [0, m(K, w)] \) yields

\[
\dot{m} = \left( \frac{1}{\alpha - \alpha^* + \gamma} \right) \left( \gamma \int_{K+\Delta}^{K} L_2(s)ds + \frac{(\rho + L_2(K + \Delta))\mu}{w - \mu} \right) > 0
\]

**Proof of Proposition 4:**

(a) The argument is similar to that for Proposition 3a.

(b) In the neighborhood of the steady state where knowledge is being accumulated, \( K \in (K^S(w), \bar{K}) \) and the production pattern will consist of two disjoint intervals as shown in Lemma 1. The most complex good produced in the steady state will be \( m(K^{SS}, w) \), because there is no learning in the steady state. The most complex good produced in the neighborhood of the steady state will be \( m(K, w - \mu) \). Therefore, \( \lim_{K \to K^{SS}, \mu \to \mu^{SS}} = m(K^{SS}, w - \mu^{SS}) > m(K^{SS}, w) \).
References


