Abstract

We study portfolio allocation in the international financial market when investors exhibit ambiguity aversion towards assets issued in foreign locations. Each country has access to a risky technology and wants to attract capital. We characterize contracts issued by countries in such an environment. Increases in the variance of the risky production process cause countries to increase the variable payment (equity) offered to investors. On the other hand, increases in investor ambiguity lead countries to decrease the equity part of the contract and increase the fixed payment in the contract. Countries with low levels of domestic wealth issue assets with lower expected return for investors, a higher fixed payment and a lower risky payment. As a result, they are exposed to higher volatility per unit of consumption. An increase in general ambiguity aversion may explain "flight to quality" of capital.

JEL codes: F21, F34, G11, G15, D81

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1 Introduction

In this paper, we ask a series of related questions on international portfolio allocation. Suppose that investors are ambiguity averse, and cannot assess well the probabilities of various events in foreign economies that issue risky securities. For example, investors may be unsure about what is the proper distribution from which production shocks are drawn. In such an environment, they may
not invest as eagerly abroad as in the domestic economy and it is well known that this may cause a home bias to arise (i.e., Uppal and Wang (2003) or Benigno and Nistico (2011)). Empirically, for example Gelos and Wei (2005) show that funds invest less in less transparent countries. What does this imply for the asset structure when issuers take this into account? What can be said then about international risk sharing, investment, and asset holdings from this perspective? How does the international asset structure change when countries are symmetric in the wealth of their resident investors and when they are not?

Questions about the international asset structure are not new and are of great importance. For example, despite the fact that developing countries are perceived as being more volatile, they cannot obtain insurance to smooth their consumption shocks. In fact, capital flows seem to be rather procyclical to emerging markets (Kaminsky et al. (2004)). Countries often issue foreign-denominated fixed-rate bonds (especially of short maturity) even if they have what Reinhart et al. (2003) call "debt intolerance", i.e. run into sovereign debt crises under debt burdens significantly lower than developed countries. For example, Reinhart and Rogoff (2009) conclude that for many emerging markets the risk of a credit event increases dramatically after a threshold of $30 - 35\%$ of external debt to GDP is reached. There is a long-lasting concern that too much of international financing relies on fixed-rate debt (for example, Lessard and Williamson (1985), Rogoff (1999), Borensztein et al. (2004) or Reinhart and Rogoff (2009)). Indeed, Figure 1 shows that the share of debt in all public and private liabilities of capital importers (countries with negative net foreign assets) is markedly higher than that of capital exporters in the Lane and Milesi-Ferretti (2007) data (and their subsequent updates).\footnote{Debt and equity liabilities of all respective countries were summed up to obtain these figures.} This is not necessarily driven by emerging countries in the sample. For the Euro zone countries for the 5 year period 2002-2006 (when the Euro was in circulation and the foreign exchange risk between the zone countries was small): capital importers\footnote{Countries that had significant net negative asset positions: Austria, Finland, Greece, Italy, Portugal and Spain.} had a debt/equity liabilities ratio on average of 2.95 (a median of 2.80) whereas among the remaining countries\footnote{Belgium, Ireland, France, Germany, Luxembourg and the Netherlands.} this average was at 1.56 (a median of 1.58). If one considers only sovereign securities very few countries offer anything else than (especially fixed-rate) debt.

We address these questions in a simple setup: countries have access to a risky production technology while private investors own productive capital that they would like to invest. Countries issue contracts in a competitive market and promise to investors a share in the risky outcome and a fixed payout. Investors, upon observing the contracts on offer, make their portfolio choices. Both the countries and investors are risk averse, and want to maximize their utility of consumption which is derived from the residual of the payout to investors and that of proceeds from capital respectively; however, investors also display ambiguity aversion towards foreign-issued assets. The problem is static with no (sovereign) default. We analyze the problem imposing different constraints on investor wealth and the worldwide capital ownership distribution.
A crucial assumption as in Benigno and Nistico (2011) (who use the approach of Hansen and Sargent (2005) and Hansen and Sargent (2007) of model misspecification to study the home bias puzzle) is that investors have different beliefs about the characteristics of foreign and domestic assets. Here we simply assume that for both home and foreign private investors the asset offered by the country follows the same (reduced) distribution, foreign investors perceive the asset as ambiguous whether home investors do not. On the operational side, we use the smooth ambiguity model proposed by Klibanoff et al. (2005), in a specification provided by Gollier (2011). These ingredients allow us to have a tractable model that can be solved and analyzed analytically.

In our model, each country has a monopoly on its production process, but faces a tradeoff: as it promises more of a fixed payout, its consumption becomes more volatile. The higher the share in the proceeds of the production process offered, the lower the risk that the country carries, but also the lower the potential proceeds. We find that increased riskiness of the production process causes countries to offer contracts with a higher risky share - they seek effectively insurance from investors that are better able to diversify risk (by investing in many noncorrelated assets). Increases in ambiguity aversion, however, cause countries to lower the floating part and increasing the fixed part, therefore insuring investors.

We also show that a country may want to issue different contracts depending on the wealth of its home investors. Countries with (relative to others) low levels of investor wealth issue contracts with lower expected return, higher fixed payouts and lower variable parts. This causes that a country’s consumption becomes more volatile per unit of consumption as they promise higher fixed payouts to investors. Hence, we offer a different story about why developing countries (i.e., associated with low investor wealth) may have more volatile consumption streams. Moreover, increases in general ambiguity aversion (but not risk) can generate “capital flight” - and increase the home bias.

If we interpret the fixed payouts as bonds while the variable parts as equity, the model can reproduce the patterns observed in Figure 1: capital importers would on average issue more debt. Moreover, countries with less domestic-based capital will issue contracts with lower expected return, and hence attract less capital. Even with a lower installed capital stock, the marginal effective product of capital can be thus lower in countries with fewer domestic investors but not because the marginal return on physical capital is low (as for example in Matsuyama (2004)). In a world with ambiguity aversion thus we can have equalized marginal returns of capital as measured by Caselli and Feyrer (2007), yet the output gains from reallocating capital can still be huge. And yes, the financial market imperfections stemming from ambiguity aversion may play a major role in preventing capital flows from "rich" to "poor" countries.

**Literature review**

There are some works that study portfolio allocations under ambiguity, but neither considers the supply of assets. An early study of portfolio choice under ambiguity, using the multiple priors approach of Gilboa and Schmeidler (1989), is by Dow and da Costa Werlang (1992). Another
one is Uppal and Wang (2003) who investigate intertemporal portfolio choice when investors take into account model misspecification. "High" ambiguity in their definition leads to portfolio underdiversification. Gollier (2011) applies the Klibanoff et al. (2005) smooth ambiguity to portfolio choice and studies the general properties of the problem with one risky and one ambiguous asset. If there is ambiguity aversion, then fixed-rate bonds would be the best solution to insure uncertain investments. Mukerji and Tallon (2004) study why agents would prefer non-indexed bonds even if they are ambiguity averse. They claim that with inflation-indexed bonds, investors are still not covered from the relative price risk in the bundles that they consume.

In the traditional home bias literature, Gehrig (1993) first incorporates an assumption that investors from different countries have different information sets. He assumes that foreign securities are considered as more risky by expected utility maximizers and derives a home bias in equities. This line of modeling is pursued also by Brennan and Cao (1997) and Kang and Stulz (1997). Nieuwerburgh and Veldkamp (2009) discuss a model where potential investors can learn additional statistical properties of various assets at home and abroad by incurring some costs. Investors that have better information about locally-issued assets prefer to learn more about their home assets because they profit more from information that the others don’t know. Learning amplifies the initial information asymmetry. The asset structure in their model is exogenous. There is empirical evidence in favor of this and other models assuming informational frictions. For example, in Andrade and Chhaochharia (2010) past U.S. FDI positions in a country are found to be positively correlated with larger stock portfolio engagements at a later period.

A related paper is Caballero and Krishnamurthy (2008) that uses the Gilboa and Schmeidler (1989) formulation of ambiguity to analyze a flight to quality episode and central bank responses.

There are several papers in the international economics literature such as Alfaro and Kanczuk (2010) or Alfaro and Kanczuk (2009) studying the macroeconomic tradeoffs of issuing various type of (fixed versus floating, short vs. long term) debt or GDP-linked securities.

**Organization of the paper.**

The paper is constructed as follows. First, in Section 2, we lay down the assumptions of our analysis, and we develop the investor’s and countries’ problem. Next, we discuss a case when investors wealth constraint is nonbinding - they remain with capital after investing in all available risky assets. Then in Section 4 we discuss portfolio decisions when the wealth constraint is binding. Section 6 concludes. Proofs are relegated to the Appendix.

## 2 Setup

We analyze the following setup. First, $M$ countries communicate contract terms of hiring capital that they will irrevocably honor to investors. They do this in a noncooperative way. Investors
observe these contract terms and make their portfolio decisions. Then capital is transferred, productivity draws are realized, output produced and payouts and consumption take place.

A typical country $j$ agrees to issue a contract containing two elements. The first, $v_j \in [0, 1]$, describes the share of the proceeds (or participation in losses) from a risky project with a stochastic return $x_j$ ran by the country. The second is a riskless return (or payment demanded) of $R_j$.\textsuperscript{5}

There are $N$ investors, each of which resides in a single country, and each the same productive capital, or wealth $w \geq 0$. An investor $l$ residing in will allocate a fraction of wealth $0 \leq \alpha_{lj} \leq 1$, into assets issued by country $j$, with $\sum_j \alpha_{lj} \leq 1$.\textsuperscript{7} An investor can choose not to invest all or only a fraction of his capital. In that case, his return on the uninvested capital is zero.\textsuperscript{8} The value of an investor’s portfolio is then $w \sum \alpha_{lj} (v_j x_j + R_j)$. Let $N_j \geq 0$ be the number of investors residing in country $j$ (so $\sum N_j = N$).

Investors often feel surer about their judgements concerning their own country than about those concerning foreign countries. In the light of recent advances in decision theory, this intuition can be translated by the fact that investors experience more ambiguity with respect to events concerning foreign countries (such as the realisation of the stochastic return $x_j$) than events concerning their home country.\textsuperscript{9} If investors have a non-neutral attitude to ambiguity, then the difference in ambiguity may have effects on investment behavior.

To capture these intuitions, we adopt the decision model proposed by Klibanoff et al. (2005). Under this model, an uncertain asset $x$ (corresponding, for example, to a stochastic return) for which there is a single (“known”) distribution can be thought of as unambiguous. By contrast, uncertain assets may be associated with several possible distributions or priors, with a second-order distribution over the possible distributions of $x$, which can be thought of as capturing the decision maker’s information about the distribution; in this case, the asset may be treated as ambiguous by the decision maker.\textsuperscript{10} Letting $\pi$ denote the priors over the returns of the asset, and $\mu$ denote the second-order distribution, decision makers choose assets $x$ to maximise:

$$V(x) = \int \Delta \varphi \left( \int_S u(x) d\pi \right) d\mu = E_\mu \varphi(E_\pi(u(x)))$$ (1)

where $u$ is a standard von Neumann-Morgenstern utility function representing the decision maker’s

\textsuperscript{4}This restriction is not necessary in the cases that we shall consider – in equilibrium it will always be true that $v_j \notin [0, 1]$.

\textsuperscript{5}Modeling sovereign default does not allow us to obtain tractable closed form solutions. To incorporate this notion, we can think that a bond contract issued by a country implicitly has $v \neq 0$ in the spirit of Grossman and Huyck (1988). There is also a considerable home bias in bond holdings as shown by Coeurdacier and Rey (2011).

\textsuperscript{6}The setup could be interpreted as if in effect the country chooses the terms on a plain-vanilla riskless bond and a risky GDP-linked security as suggested for example by Schiller (1993).

\textsuperscript{7}Therefore, we rule out any short selling. In some equilibria that we analyze this will not be a binding constraint.

\textsuperscript{8}So, we either assume that he has an opportunity to invest in some risk-free asset yielding zero or he let’s his capital sit idle. Note that, given the use of CARA utility functions, the solutions are the same as if investors had the opportunity to invest in a risk-free bond (after appropriate normalisation).

\textsuperscript{9}This is conceptually different than assuming arbitrarily that foreign assets are more risky as in Gehrig (1993)).

\textsuperscript{10}Whether or not the asset is actually ambiguous depends on the precise shape and interaction between the distributions. The definition of ambiguous assets is a currently debated question in the decision-theoretic literature; see Klibanoff et al. (2005), but also, for example ...
risk attitude, and \( \varphi \), sometimes called a “transformation function”, represents the decision maker’s ambiguity attitude (in the sense made precise in (Klibanoff et al., 2005, §3)).

To obtain a tractable model, we draw on the specification used by Gollier (2011). Each investor has constant absolute risk aversion, and hence a utility function of the form \( u(z) = -(1/\theta) e^{-\theta z} \) where \( \theta > 0 \) is the risk aversion. Each investor is assumed to be ambiguity averse and indeed to have constant relative ambiguity aversion; the transformation function is thus of the form \( \varphi(U) = -\left(\frac{U^{1+\gamma}}{1+\gamma}\right) \) where \( \gamma \geq 0 \) is the ambiguity aversion. Risk aversion and ambiguity aversion are assumed to be same for all investors.

To represent ambiguity, first consider an investor foreign to country \( i \). The only priors over the stochastic return \( x_i \) that he considered possible are normally distributed with the same variance \( \sigma_i^2 \) though different means \( m \); that is, he considers plausible only distributions \( \tilde{x}_i \sim N(m, \sigma_i^2) \). His second-order prior over this set of distributions is itself normally distributed, with mean \( \mu_i \) and variance \( \delta_i \sigma_i^2 \): \( \tilde{m}_i \sim N(\mu_i, \delta_i \sigma_i^2) \). In other words, foreign investors are sure that the variance of the stochastic return is \( \sigma_i^2 \), but are unsure about the mean; where this uncertainty (or ambiguity) is represented by a distribution over the possible values of the mean. Note that a expected utility decision maker, on receiving subjective information corresponding to the two-stage distribution just described would collapse or reduce the second-order distribution; it is straightforward to see that he would then choose according to a normal distribution on \( x_i \) with mean \( \mu_i \) and variance \( (1+\delta_i)\sigma_i^2 \). We assume that an investor whose home country is \( i \) chooses in just this way: he has a single prior for the stochastic return \( x_i \) that is normally distributed with mean \( \mu_i \) and variance \( (1+\delta_i)\sigma_i^2 \), and a degenerate second-order prior that puts all probability weight on this distribution. As noted, home and foreign investors have the same “reduced” distribution for the stochastic return \( x_i \); the difference is that whilst home investors have a one-stage distribution, foreign investors have two-stage (or second-order) uncertainty. This represents the fact that for the latter, but not for the former, the stochastic return is ambiguous.

Once a country \( i \) obtains capital \( k_i \) on the market, it invests it in a risky production process. The proceeds can be described by \( y_i = x_i f(k_i) \) where \( f(k_i) = k_i \) and \( x_i \) is stochastic productivity that is unknown prior to investment. We assume that the country has the same knowledge as home investors; that is, that it treats the stochastic return as being distributed according to the Normal distribution with \( x_i \sim N(\mu_i, (1+\delta_i)\sigma_i^2) \). Finally, we assume that productivity draws across countries are identically and independently distributed (so, in particular, the \( \mu_i \) and \( (1+\delta_i)\sigma_i^2 \) are the same).

To be able to solve the model, we need to assume that both the country and the investors can have negative consumption and that the country can make payouts to investors even if negative productivity is realized. Given investor and country preferences that admit negative consumption and their usage in the finance literature we find the assumptions on the productivity generating process to be awkward but not inadmissible.
2.1 The sovereign’s problem

The sovereign chooses contract terms so as to maximize its expected utility from consumption. It has an access to a risky technology. The country is risk averse (or, as above not ambiguity averse towards its own productivity process), and has an exponential utility of consumption with the degree of absolute risk aversion of $A$. The sovereign understands that the capital that can be raised in the international capital market will be a function of the contract terms offered by all countries (denote it by a matrix $C = [v, R]$ where $v$ and $R$ are vectors of contract terms) and the world distribution of investor wealth $wN$. Therefore, the problem of the country $i$ can be written as that of choosing the contract terms $v_i$ and $R_i$ so as to maximize the following utility\(^\dagger\).

\[ -(1/A) \int e^{-A[(1-v_i)xf(k(C,wN))-R_i]}(1/\sqrt{2\pi(1+\delta_i)\sigma_i})e^{-((x-\mu_i)^2/(2(1+\delta_i)\sigma_i^2))}dx \] \hspace{1cm} (2)

Plugging in for $f(\cdot)$, this problem of expected utility maximization can be rewritten as

\[ -(1/A) e^{A[R_i-(1-v_i)\mu_i]}(1+\delta_i)\sigma_i^2(1-v_i)^2/(2(1+\delta_i)\sigma_i^2) \] \hspace{1cm} (3)

Note that the country can offer a contract $(0, 0)$ and therefore obtain a utility of at least $-(1/\theta)$.

2.2 Investor’s problem

First, we can obtain the expression for expected utility for each investor $l$, domiciled in country $i$ under the assumption that $x_j \sim N(m_j, \sigma_j^2)$ just as in Gollier (2001).

\[ U_l(\alpha) = -(1/\theta)e^{-\theta w \sum_j [\alpha_{ij} v_j m_j + R_j] - ((\theta w \sigma_j^2)/2)v_j^2 \alpha_{ij}^2} \] \hspace{1cm} (4)

This is the inside integral in equation (1); plugging it into that equation under the specification set out above yields:

\[ V_l(\alpha) = -(1/\theta)e^{-\theta w [\alpha_{i1} v_i + \mu_i + R_i] + \theta (1+\delta_i)(\sigma_i^2 \alpha_i^2 v_i^2)/2 - \theta w \sum_{j \neq 1} [\alpha_{ij} v_j m_j + R_j] - (\sigma_j^2 \theta w(v_j \alpha_j)^2)(1+\delta_j + \delta_j)^2]/2 \] \hspace{1cm} (5)

From (5) optimal portfolio decisions can be derived under the constraints that $1 \geq \alpha_{ij} \geq 0, \forall j$ and $\sum_j \alpha_{ij} \leq 1$.

With exponential preferences, given a choice between several risky assets and one safe asset, investors place some constant wealth in each of the risky securities (see chapter 13 of LeRoy and

\[ ^{\dagger}In what follows, simple contracts such as $v_i = 0$ and $R_i > 0$ turn out not to be an equilibrium outcome. Countries would then engage in Bertrand-competition like bidding for capital. As they are risk averse, however, they might not want to obtain the entire capital in the world but would limit the quantity of bonds offered at some rate. The potential of a country to offer a contract $v_i = 0$ and $R_i > 0$ would also mean that each sovereign can offer a risk-free bond, which is a strong assumption.\]

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Werner (2001)) if short-selling is available. As we rule out short selling, we shall consider two cases. The first one will be if investors can satisfy their optimal purchases with the wealth on hand (which we shall call the “unconstrained” case). The other, when investors exhaust all their wealth by investing only in risky assets – the “constrained” case. The latter case is more realistic as capital does not sit idle; the former however is much easier and tractable to solve and interpret.

3 The “unconstrained” case

Here we consider the situation when some of the investor’s wealth is going to remain uninvested, i.e. in equilibrium \( \sum_j \alpha_{lj} < 1 \) and this is going to be true for all \( l \). Then, from the investor’s problem of maximizing (5) we obtain for assets issued domestically (i.e. in home country \( i \))

\[
\alpha_{li} = \frac{[v_i \mu_i + R_i]}{\theta v_i^2 (1 + \delta) \sigma_i^2}
\]

and for any foreign issued asset by some country \( j \)

\[
\alpha_{ij} = \frac{[v_j \mu + R_j]}{\theta v_j^2 [(1 + \delta) + \delta \gamma] \sigma_j^2}
\]

Neither the contract terms offered by other countries nor the world wealth distribution does not matter for asset demands in this case. Note that any risky asset with a positive expected return will find positive demand.

Then, a country \( i \) issuing an asset with characteristics \((v_i, R_i)\) can obtain

\[
k_i = \left[ N_i \frac{[v_i \mu + R_i]}{\theta v_i^2 (1 + \delta) \sigma_i^2} + \sum_{j \neq i} N_j \frac{[v_j \mu + R_j]}{\theta v_j^2 [(1 + \delta) + \delta \gamma] \sigma_j^2} \right]
\]

\[
= \frac{[v_i \mu + R_i]}{\theta v_i^2 \sigma_i^2} \left[ \frac{(1 + \delta) \sum N_j + \delta \gamma N_i}{(1 + \delta)((1 + \delta) + \delta \gamma)} \right]
\]

Maximizing (3), and rewriting the first order conditions we obtain

\[
\mu - \frac{\partial k_i}{\partial \mu} - \sigma^2 \theta (1 - v_i) k_i = 0
\]

\[
R_i = (1 - v_i) \mu_i - \frac{k_i}{\partial R_i} - \sigma^2 \theta (1 - v_i)^2 k_i
\]

Evaluating (9) and (10) we obtain

\[
v_i^* = \frac{((1 + \delta) N + N_i \delta \gamma)}{\left(2 \frac{\theta}{\delta \gamma} (1 + \delta) + \delta \gamma + ((1 + \delta) N + N_i \delta \gamma) \right)}
\]

\[\text{Recall that } N = \sum N_j.\]
\[ R_i^* = -\frac{v_i^* \mu}{2} \]  

Therefore, the expected return from a contract \( E[v_i x_i + R_i] = \frac{v_i^* \mu}{2} \) while the variance is \( (v_i^*)^2 (1 + \delta)\sigma^2 \).

### 3.1 Discussion and comparative statics

First of all, note that equation (11) can be rewritten as follows:

\[ v_i^* = \frac{1}{2\frac{\theta}{AN} \left( \frac{1+\epsilon \gamma}{1+n_i \epsilon \gamma} \right) + 1} \]  

where \( n_i = N_i/N \) is the proportion of the investor population domiciled in country \( i \), and \( \epsilon = \frac{\delta}{1+\delta} \) can be thought of as the proportion of the foreign investor’s uncertainty which is in the form of ambiguity. Note that \( 0 \leq \epsilon \leq 1 \): at \( \epsilon = 0 \), the second-order prior is degenerate and there is no ambiguity, and as \( \epsilon \) tends to 1, the second-order uncertainty completely dominates the first-order uncertainty.

To interpret this formula, first consider the simple case of one investor (whose home country is the country in question). Then the optimal contract offered by the country is \( \frac{1}{2\frac{\theta}{AN} \left( \frac{1+\epsilon \gamma}{1+n_i \epsilon \gamma} \right) + 1} \). This can be understood as follows: as the investor’s risk aversion (\( \theta \)) increases, he buys less of the risky asset and to compensate this the country reduces the risky share of the contract offered (the \( v \)), at the cost of bearing more risk itself. On the other hand, as the country’s risk aversion (\( A \)) increases, it is less willing to bear the risk itself, and so tries to push more of it onto the investor, by raising \( v \).

The interpretation of the general case (equation (13)) can be given by considering that, in the presence of ambiguity, the country acts as if it is facing a single representative investor whose risk aversion is \( \frac{\theta}{N} \left( \frac{1+\epsilon \gamma}{1+n_i \epsilon \gamma} \right) \). The consequences of changes in risk, risk aversion, ambiguity, ambiguity aversion, world wealth and proportion of world population in the country can be read off almost directly:

- An increase in risk aversion of each investor \( \theta \) increases the “aggregate” risk aversion, and, as noted already, this causes the country to insure the investors more, by offering a lower risky share in the contract (\( v \)).

- Recall that \( \epsilon \) can be thought of as the proportion of a foreign investor’s uncertainty that is in the form of ambiguity. In particular, it is the ratio between the extra variance term added per unit of ambiguity aversion (\( \delta \sigma^2 \)) and the variance experienced by an individual that reduces the second order uncertainty ((1 + \( \delta \))\sigma^2). Hence, if one identifies the risk of the stochastic process with the variance of the reduced distribution, and the ambiguity with the variance of the second-order distribution,\(^{13}\) then an increase of risk (with constant ambiguity)

\(^{13}\)Bla blabla about how this is not such a bad assumption.
corresponds to a decrease in $\epsilon$, and an increase in ambiguity (with constant risk) corresponds to a increase in $\epsilon$. Since $n_i \leq 1$, increasing $\epsilon$ causes the representative agent’s risk aversion to rise (or not to fall, if $n_i = 1$), and hence induces lower levels of $v$ for the reasons explained above, whereas decreasing $\epsilon$ imply a lower risk aversion for the representative agent, and hence induces higher values of $v$.

Risk and ambiguity thus have effects that go in opposite directions. An increase in risk (for constant ambiguity) means that the population act as if they are less risk averse, because the proportion of ambiguity is weaker; the country needs to insure investors less on the aggregate and so increases the risky share. By contrast, an increase in ambiguity (for constant risk) means that the population acts on the aggregate as if they are more risk averse; thus the need for more insurance, and lower $v$.

- An increase in ambiguity aversion has a similar effect to an increase in $\epsilon$, given the form of the equation. In other words, as ambiguity aversion increases, the representative agent acts as if it is more risk averse, and hence the country offers contracts with lower $v$.

- Most importantly, the proportion of investors in the home country has an important effect on the optimal contract. As could be expected, as $n_i$ increases, the representative investor acts as if he is less risk averse. This of course makes sense: as $n_i$ increases, fewer investors perceive the contract offered by the country as ambiguous. Note that this is a crucial difference from a model involving only risk aversion and no ambiguity ($\epsilon = 0$): since investor’s risk aversion is the same for assets in all countries, it affects demands for assets in all countries equally. By contrast, in the current model, different countries may offer different contracts because the number of investors in the home country, and hence the number of investors considering the contract offered to be ambiguous can be different, even if the investors’ risk (and ambiguity) attitudes are the same. Hence the effective cost of capital for a country with few domestic-based investors is higher than for wealth-abundant countries. Note that the phenomenon does not arise from a difference in the quantity of information: in reduced form, all investors are using the same distribution. At best, it is a difference in the quality of information: whereas home investors integrate all the information together (reducing everything to single first-order distribution), foreign investors keep the second-order and first-order parts separate.

Note that, by equation (12), in each case, the constant part of the contract moves in the opposite direction from the risky part. Given (11) and (12) the expected return offered to investors changes similarly as the risky part. Increased risk and ambiguity have thus a divergent impact on the expected return offered to investors. Increased productivity risk causes the country to offer a higher expected return as investors need more incentives to hold risky assets rather than the safe asset yielding zero. Increased ambiguity, however, incites the country to lower the expected reward on invested capital. Ambiguity therefore depresses equilibrium asset returns offered by countries.
– the effect comes not only from low investor demand\textsuperscript{14}.

Moreover, one could ask about the effect of risk, ambiguity, ambiguity aversion and the proportion of home investors on the capital invested in country $i$. We have that:

$$k_i = \frac{\mu}{(1+\delta)\sigma^2} \left( 1 + \frac{N}{2\theta} 1 + \epsilon \gamma n_i \right) \left( \frac{1 + \epsilon \gamma}{1 + \epsilon \gamma n_i} \right)$$ \hspace{1cm} (14)

Country $i$’s choice of $v_i$ and $R_i$ can itself be thought of a choice of portfolio, since, via $k_i$ it implies that the country will face a stochastic consumption of $(1 - v_i)k_i x_i - R_i k_i$. If the country could choose how much $k_i$ it would like to invest in the technology, it is straightforward to check that the optimum is $\frac{\mu}{(1+\delta)\sigma^2 A}$. The actual capital that the country manages to attract is this plus a correction term that is inversely proportional to the risk aversion of the representative agent $1 \left( \frac{1 + \epsilon \gamma}{1 + \epsilon \gamma n_i} \right)$. To ascertain the effect of increased ambiguity, at a fixed level of risk, it suffices to compare differences in $\epsilon$ with $(1 - \delta)\sigma$ kept fixed. It is clear (for reasons similar to those mentioned above), that as ambiguity increases, the capital attracted by countries decreases (unless $n_i = 1$, i.e. the world’s wealth is concentrated in the country). Similarly, as the proportion of investors at home decreases, the capital invested in the country decreases. Moreover,\textsuperscript{15}

$$\frac{\partial^2 k_i}{\partial n_i \partial \epsilon} = \frac{\mu}{(1+\delta)\sigma^2} \frac{N}{2\theta} \frac{\gamma}{1 + \epsilon \gamma} > 0 \hspace{1cm} (15)$$

so the negative effect of an increase in ambiguity is stronger for countries with fewer home investors. As above, if one considers what happens as the proportion of the uncertainty treated as risk increases (with the total uncertainty being kept fixed), then the tendencies are in the opposite direction (this corresponds to a decrease in $\epsilon$ with $(1 - \delta)\sigma^2$ fixed). As the proportion of risk increases, the capital attracted by country increases, and this increase is higher for countries with fewer home investors.

Nevertheless, the variance of any countries’ consumption irrespective of the number of investor residents remains $\frac{\mu}{(1+\delta)\sigma^2 A}$, so countries with less wealthy investors do not experience a higher variance of consumptions than their more wealthy peers in this “unconstrained” case.\textsuperscript{16} They do, however, face a lower expected consumption stream. Therefore, they face a higher variance per unit of consumption (GNP).

Finally, let us consider how the proportion of the country’s home and foreign investment is affected by ambiguity. To that end, let us consider the proportion of capital which comes from home investors, $\eta_i = N \frac{\eta_i}{\frac{\mu}{\mu+R_i} / \theta / (1+\delta)\sigma^2 k_i}$. Calculating, we have:

$$\eta_i = \frac{n_i (1 + \epsilon \gamma)}{(1 + \epsilon \gamma n_i)} \hspace{1cm} (16)$$

\textsuperscript{14}Given our assumption that $f(k_i) = k_i$ so that the marginal productivity of capital is constant.

\textsuperscript{15}Assuming $(1 - \delta)\sigma^2$ fixed.

\textsuperscript{16}Given their preferences, both countries and investors exhibit quasilinear preferences towards risk. This is why investors always want to put a fixed amount in each risky asset, and the country selects capital so that it is the mean of their consumption that is affected while selecting their preferred risk.
From which it simply follows that, as the proportion of investors resident in the country rises, then the proportion of capital which comes from home investors rises. Moreover, as the ambiguity or the ambiguity aversion rises, the proportion of capital which comes from home investors rises. This is intuition, as the foreign investors are more pessimistic, either because they perceive more ambiguity or because they are more ambiguity averse, the country relies more on home investors, that are not affected by ambiguity. Note that whilst the total capital invested depends on the total amount of uncertainty \((1 + \delta)\sigma^2\), the proportion of capital coming from home investors is independent of the total amount of uncertainty: it only depends on the proportion of uncertainty that is ambiguity rather than risk.

Note finally that an investor foreign to country \(i\) invests:

\[
\mu \frac{1}{(1 + \delta)\sigma^2} \left( \frac{1}{AN(1 + n_i \epsilon)} + \frac{1}{2\theta(1 + \epsilon)} \right)
\]

in the asset offered by \(i\). Hence, each investor invests more in a foreign country with a low proportion of home investors than in a foreign country with a higher proportion of home investors. This is explained by the fact that countries with lower proportions of home investors have to raise more of their capital abroad, so offer terms that are more attractive to foreign investors.

3.2 Adding moral hazard

Some authors view the problem of sovereign debt as involving moral hazard on the side of the borrowing country. For example, the country may not exert enough effort to increase the success chances of the investment. Indeed, if an agent has a risky project and wishes to borrow capital, there might be a preference for using debt contracts over equity contracts to provide proper incentives.

We incorporate thus a possibility that the sovereign can exert effort to influence ex ante the productivity level to investigate how this would affect our base results. Let the cost of effort \(\xi_i\) affecting the mean \(\xi_i \mu\) and the variance of the process \((1 + \delta_i)\xi_i^2 \sigma_i^2\) be given by a concave function \(Z(\xi_i)\). The timeline is as follows. The sovereign chooses \(v_i\) and \(R_i\). Investors observe this, and make investment choices taking into account what the country will choose next as its effort level \(\xi_i\). Then, the country chooses \(\xi_i\).

The country’s maximization problem of choosing the optimal \(x_i\) is then in the last step \(^{17}\)

\[
-\frac{1}{A}e^{A[R_i^* - (1-v_i^*)\xi_i \mu | k_i + (1+\delta_i)\epsilon_i^2 \xi_i^2 (1-v_i^*)^2 \xi_i^2 + AZ(\xi_i)]}
\]

As a result, the optimal \(v_i^*\) is then as in (11) while

\[
R_i^* = -\frac{v_i^* \xi_i^2 \mu}{2}
\]

\(^{17}\)In our setup, the solution for \(v\) and \(R\) is the same if the effort affects only the mean of the process or if the effort function is some \(Z(\xi_i, k_i)\).
In our portfolio choice problem, unless the country can force the domestic agents to purchase its own securities\footnote{Episodes of dollarization or runs on the currency attest that countries cannot easily force people to hold their assets. Alternatively, one can use arguments as in Broner and Ventura (2011) why countries cannot discriminate between home and foreign investors.}, the moral hazard in investment would not affect the contract structure if there were no ambiguity aversion. Even if investors exhibit ambiguity aversion, the choice of $v_i^*$ is unaltered. If the effort cost is $Z(\xi_i)$ then the optimal choice of effort $\xi_i^*$ is the same for each country and in particular it does not depend on the capital owned by domestic investors. Therefore, if $\xi_i^*$ is the same across countries, then it does not affect $R$ differently for different countries. Consequently, it does not affect our basic conclusions.

### 3.3 Adding exchange rate risk

We have sidestepped so far the issue about the currency composition of the assets that are issued by the country. This may be of importance to foreign investors: securities with yields denominated in foreign currencies may carry additional risk.

Bonds denominated in foreign currencies (but also any GDP-linked securities) are a guard against moral-hazard induced inflation that could be stoked by the borrowing country in order to diminish its real obligations after any bonds are bought by investors. One solution to this moral hazard problem would be to offer a higher participation $v$ than without such an inflationary risk. However, if the Purchasing Power Parity holds (and there is no stochastic risk in its movement), then any increase in inflation ex post would be matched by a corresponding change in the exchange rate, leaving the real returns unchanged to either domestic or foreign investors holding such assets. If the country chose to issue only foreign-denominated assets then it effectively would face the problem just like we initially considered.

A different issue is when the exchange rate follows some stochastic process and there is no moral hazard problem. In principle such exchange risk by itself could be totally hedged by investors. Moreover, in the home bias literature the standard arguments why exchange rate risk would matter come out as an unconvincing reason empirically (van Wincoop and Warnock (2010) Baxter and Jermann (1997)) for the existence of the home bias in equities.\footnote{A home bias in investment was found also where no exchange risk is involved as witnessed by the fund data analyzed by Coval and Moskowitz (1999) and Coval and Moskowitz (2001). These authors ascribe the home bias to informational asymmetries.}

Here we extend the model to tackle at least partially the question how a country would behave if foreign investors suffered exchange rate risk whilst investing in these countries’ assets. A natural way to model this would be to consider an additional stochastic process that affects the returns from the investments abroad. Unfortunately, we were unable to find such a process that could yield closed-form solutions.\footnote{Assuming that the exchange rate process is also normally distributed would yield a product-normal distribution after which no closed-form solutions were found.} For simplicity we consider thus an exchange rate risk as being an independent zero-mean normal process with variance $\tau^2$ affecting the capital invested $k$ and how
this changes the contracts issued by the sovereign. We study a setup in which the country issuing assets insures fully foreign investors on the exchange rate risk that they may be facing. Hence, the maximization problem of the country (3) becomes

\[ -\frac{1}{A}e^{A[R_i-(1-v_i)\mu]k(C,wN) + (1+\delta_i)(1+\delta_i)\sigma^2(1-v_i)^2(k(C,wN)) + \tau^2A^2v_i^2(I,C,wN)} \]

(20)

where \( k_F(C,wN) \) is the capital obtained from foreign investors. The solution for \( v_i^* \) is as in (11) and for \( R_i^* \) it is

\[ R_i^* = -v_i^*\mu-
\]

\[ \frac{v^* + \tau^2A(1+\delta_i)(1+\delta_i)\sigma^2(1+\delta_i)\gamma(\sum X_j)^2)}{2v^* + \tau^2A(1+\delta_i)(1+\delta_i)\sigma^2(1+\delta_i)\gamma(\sum X_j)^2} \]

(21)

With higher exchange rate risk \( \tau^2 \) the interest rate on issued bonds is lower.

4 The “constrained” case

Investor’s shadow value of capital in the unconstrained case was zero. A more realistic and interesting scenario is when the investor is constrained in his choices by his wealth or can invest only in the risky assets to which we turn now. Countries now need to compete with one another to attract international capital. We simplify assuming one investor in each country and setting \( A = \theta \).

Imposing such restrictions on the programming problem (5) we obtain the shares of wealth

\[ \alpha_{ii} = 1 - \sum_j \frac{(v_i \mu + R_i) - \sum_j (v_i \mu + R_j)}{\sum_j (1+\delta_j)\sigma^2 + \delta \gamma \sigma \sum X_j} \]

(22)

\[ \alpha_{ij} = \frac{(v_i \mu + R_i) - \sum_j (v_i \mu + R_j)}{\theta w_i v_j^2 ((1+\delta)\sigma^2 + \delta \gamma \sigma^2) + v_j^2 ((1+\delta)\sigma^2 + \delta \gamma \sigma^2)} \]

(23)

Then, a country \( i \) issuing an asset with characteristics \((v_i, R_i)\) can obtain

\[ k_i = \]

\[ \begin{bmatrix}
    \frac{w_i - \frac{1}{\theta} \sum m \frac{v_i \mu + R_i - v_m \mu - R_m}{\gamma^2((1+\delta)\sigma^2 + \delta \gamma \sigma^2)}}{\sum_j (v_i \mu + R_i - v_m \mu - R_m)} \\
    + \frac{1}{\gamma^2((1+\delta)\sigma^2 + \delta \gamma \sigma^2)} \sum_j (v_i \mu + R_i - v_j \mu - R_j) \\
    + \frac{1}{\gamma^2((1+\delta)\sigma^2 + \delta \gamma \sigma^2)} \sum m \neq i v_m^2 (1+\delta)\sigma^2 \\
\end{bmatrix} \]

(24)

4.1 Symmetric countries with identical investor wealth, \( w > 0 \)

We first solve and analyze the easiest case, when all countries are identical and have each an investor with wealth \( w > 0 \).
Maximizing (3), from the first order conditions (9)-(10), imposing symmetry and using the notation above we obtain
\[ v^* = \frac{(1 + ne\gamma)}{1 + 2n + 3ne\gamma} \] (25)
and
\[ R^* = (1 - v) \mu - \left( \frac{(1 + ne\gamma)(1 + e\gamma)}{(N - 1)(1 + 2ne\gamma)} + 4 \frac{(1 + e\gamma)^2}{N^2(1 + ne\gamma)^2} \right) v^2 (1 + \delta) \sigma^2 \theta w \] (26)
where \( n = \frac{1}{N} \).

The expected return offered to investors is then
\[ v^* \mu + R^* = \mu - \left( \frac{(1 + ne\gamma)(1 + e\gamma)}{(N - 1)(1 + 2ne\gamma)} + 4 \frac{(1 + e\gamma)^2}{N^2(1 + ne\gamma)^2} \right) v^2 (1 + \delta) \sigma^2 \theta w \] (27)

As previously, consider first what happens when there is no ambiguity aversion and no uncertainty about foreign economic fundamentals. Still, \( v^* = \frac{N}{(N+2)} \) while \( R^* = \frac{2}{(N+2)} \mu - \frac{(N^2 + 4N - 1)}{(N-1)(N+2)} \sigma^2 \theta w \).

With ambiguity aversion, the equity part of the contract that is offered is exactly the same as in the unconstrained case in (11). The difference now lies in the fixed part of the contract \( R \) which reflects the fact that countries compete now with one another for capital. More realistically, \( R \) now can be positive. The fixed part offered to investors now declines with the increase in individual investor wealth \( w \) and hence capital availability.

We obtain the same comparative statics as in the unconstrained case for the changes in \( v \) as ambiguity aversion and the variance of the technology productivity increase. \( R \) declines as risk increases, while for ambiguity, we obtain a bell shape. Nevertheless, expected investor return always declines as both ambiguity and risk increase. In this constrained case, interestingly, the expected return offered to investors always decreases with the increases both in risk and ambiguity aversion. Note however that the expected return cannot optimally fall below some threshold (at least zero) as then the investor would always do better by withholding investment and obtaining a safe zero return on his capital.

### 4.2 Tractable case with asymmetric countries: one country with an investor with wealth \( w > 0 \), other countries with no wealth.

Although we can obtain in principle closed-form solutions for any distribution of investors we can obtain solutions to perform tractable comparative statics if we assume that there is only one country with wealth \( w > 0 \) (so \( N = 1 \) and \( M > 0 \)). We obtain similar patterns for countries asymmetric in wealth as observed in the unconstrained case in Section 3: \( v_N > v_M \) and \( R_N < R_M \).\(^{21}\) It immediately follows that the bond liabilities to expected equity liabilities \( (kR)/(kv\mu) \) are higher for the countries with no wealth (as in Figure 1) than for the country with an investor. Again the

\(^{21}\)Detailed calculations available upon request.
expected return to capital is lower in the countries with lower (no) wealth. As ambiguity increases, the discrepancy between the capital invested in the "wealthy" and "poor" countries increases: \( \frac{\partial (k_N - k_M)}{\partial \epsilon} > 0 \) - so in the case of an increase of general ambiguity (but not risk) we can have "capital flight". This markedly contrasts with the case when there is no ambiguity aversion on the side of investors: then \( v_N = v_M \) and \( R_N = R_M \) so the issued contracts are the same as is the capital invested, output and consumption.

5 Extensions

There is a number of extensions and robustness checks that we wish to pursue. An important question is whether our results go thru if we assume a different formulation of ambiguity aversion, namely that from Gilboa and Schmeidler (1989). An interesting avenue to pursue is the notion that investors may be more ambiguity averse towards events further in the future; hence there could be a natural tendency for countries with relatively little domestic capital (owned by "savvy" investors) to issue short-term securities. It would be welcome to decentralize the model to analyze firm behavior and the multicountry model with "constrained" capital should be explored further. Another drawback of the model is that the productivity process is Normal-distributed - for example we can obtain some closed form solutions for special cases with Weibull-distributed productivity. It is also imperative to perform simulations to allow for different preferences towards risk and ambiguity and a fully fledged neoclassical production function with at least two factors of production to fully address the findings raised by Caselli and Feyrer (2007). Sovereigns may issue also a wide array of contracts, and try to discriminate among different investors or introduce capital controls. Also, a different moral hazard problem should be analyzed when the country could hide some of the output after the productivity shock would be realized, i.e. cheat the investors on \( v \).

6 Conclusions

We have studied the implications of ambiguity aversion on the international security design. Our conclusion is that with ambiguity aversion countries offer more fixed income securities that are not GDP-linked. There are marked differences in the contract terms and the composition of capital allocation across the world which depend crucially on the initial distribution of capital.
References


Figure 1: Ratios of debt to equity liabilities among net capital importers (solid line) and capital exporters (dashed line) for a sample of all 177 countries from the Lane and Milesi-Ferretti (2007) data set available for years 1999-2007.