The Trade and Welfare Effects of Mergers in Space

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Abstract

This paper analyzes the consequences of a merger in a spatial framework, thereby elaborating on three channels of influence: a price increase, an adjustment in plant location, and a harmonization in production costs due to a firm-internal technology transfer. In a detailed welfare analysis, we show that larger regions are better off after the merger, while smaller regions may lose if the production cost differential across firms is negligible and/or a technology transfer is not feasible. Furthermore, we provide novel insights into the trade pattern effects of a merger. In this respect, the main result of the paper is that an adjustment of plant location in space can trigger a trade reversal.

Key words: Spatial competition; cross-border merger; trade pattern; welfare analysis

JEL classification: F12; F23; L10

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1 Introduction

There is by now broad consensus among economists that cross-border mergers are a key aspect of globalization. Empirical evidence suggests that cross-border mergers are even more important than foreign greenfield investment and that their share in foreign direct investment (FDI) has considerably increased in recent years (UNCTAD, 2000).\(^1\) Despite this empirical regularity, “the theoretical literature on cross-border mergers is tiny, both in absolute terms and relative to the enormous literature on greenfield FDI” (Neary, 2007, p. 1). It is therefore not surprising that key issues in the context of cross-border merger effects are still unexplored. In particular, while the few existing studies primarily focus on the strategic motive for mergers and the consequences of trade policy\(^2\), none of them has put emphasis on the role of space.

An adequate discussion of space requires a model, in which countries are treated as areas and not as points. Tharakan and Thisse (2002) have suggested to use a variant of Hotelling’s line model as a theoretical vehicle for investigating to what extent national transport costs and intra-regional adjustments in firm location influence our understanding about globalization effects.\(^3\) Egger and Egger (2007) have extended the model to account for vertical specialization and the consequences of international outsourcing. However, both studies ignore an important feature of the recent wave of globalization, namely the change in firm organization due to cross-border (or inter-regional) mergers.

It is the purpose of this paper to link the literature on cross-border mergers to recent work on trade in a Hotelling framework. Starting point of our analysis is the long-run free trade equilibrium in Tharakan and Thisse (2002), with two differently sized regions

\(^{1}\)Furthermore, Gugler, Mueller, Yurtoglu, and Zulehner (2003) provide empirical support for an increase in cross-border mergers as a share of total mergers.


\(^{3}\)Shachmurove and Spiegel (1995) have also used the Hotelling line to study trade liberalization in a spatial framework. However, as pointed out by Tharakan (2001), the equilibrium analyzed there does not exist due to the assumption of linear transport costs.
(represented by segments of the unit interval), quadratic transport costs, and two firms located at the Western and Eastern end of Hotelling’s line, respectively. However, in contrast to the baseline model of Tharakan and Thisse (2002), we allow for differences in the production costs of the two firms. A merger in this setting gives rise to three sources of profit gains, namely (i) higher prices, (ii) changes in firm location, and (iii) a firm-internal technology transfer and the use of the best-practice technology in both production plants. While sources (i) and (iii) have been highlighted in the existing merger literature (see Neary, 2007, for an overview), the second source of profit gain points to a new channel of influence, which can only be addressed in a spatial setting.\footnote{As the Hotelling framework presumes that firms set prices (and not quantities), our analysis is related to work by Saggi and Yildiz (2006) who investigate the linkage between trade and merger policy in a setting with price competition and differentiated goods. While the literature on industrial organization points out that there is in general an important difference between models of quantity and price competition with respect to the possibility of merger gains (see Salant, Switzer, and Reynolds, 1983; Deneckere and Davidson, 1985; for two influential contributions), this distinction plays a minor role in our setting. The reason is that in our analysis a merger induces a monopolization of the market with a single producer serving all consumers \textit{ex post}. In this case, however, a profit gain of the merger is guaranteed in any standard model of industrial organization, irrespective of whether price or quantity competition is accounted for.}

A comparison between the equilibrium with independent (strategic) firm decisions and the equilibrium with joint profit maximization, gives important insights for the possible trade structure and welfare effects of a merger. With regard to the trade pattern effects, our analysis points to a non-trivial interplay of size and technology differences. In particular, we identify parameter regions, for which a merger leads to a reversal of trade, and we determine the size of these regions as a function of the cost differential between the two firms. The welfare analysis confirms the well-established result that a merger leads to a profit increase which comes at cost of a consumer surplus loss. This is a consequence of market monopolization after the merger. However, overall welfare increases due to lower transport costs after an adjustment in firm location and/or due to a more efficient production structure. The welfare increase rises with the cost differential between the two
firms. This outcome is less surprising in the case of a technology transfer. However, it also holds if a technology transfer is excluded and the two plants differ in their production costs before and after the merger. In the latter case, the integrated firm has an incentive to increase the market share of its low-cost plant through an adequate choice of location. This points to a so far unexplored channel through which a merger influences welfare: adjustments of plant location in space.

Assuming that total profit income is equally distributed among consumers, we can also derive regional welfare effects. In this respect, the main finding of our analysis is that a region tends to be worse off after the merger if it is sufficiently small and production cost differences are not too large. Again, it is the adjustment of firm location – and the associated increase of transport cost expenditures for consumers at the Eastern and Western end of the Hotelling line – which is responsible for this result. Only if cost differences are sizable and a merger leads to a technology transfer with the best-practice technology being used in both production facilities, a welfare increase is guaranteed in any region, irrespective of the prevailing size differences.

The remainder of the paper is organized as follows. In Section 2, we set up the basic model with free trade and two (independent) firms. The impact of a merger on plant location, prices, welfare and trade pattern is at the agenda of Section 3. There, we distinguish three scenarios. In the first scenario, we assume that firms do not differ in their production costs, in order to obtain a benchmark for the possible merger effects. In scenario two, we consider production cost differences but exclude the possibility of a technology transfer. The impact of a technology transfer is addressed in scenario three. A distinction between these three scenarios is useful because it allows us to separate different channels of influence and to derive a detailed picture of the possible merger effects in a Hotelling framework. Section 4 provides a short summary and some concluding remarks.
2 Basic model set-up: free trade with two firms

Consider a spatial model à la Hotelling with two producers, one operating in the West (W) and one in the East (E). Producer \( \ell \) is located at address \( x_\ell \) on a line of length one: \( x_\ell \in [0,1], \ell = W,E \). \( x_W \) is the location of the Western producer and, therefore, we have \( x_W < x_E \). Firms may differ in their marginal production costs, while fixed firm set-up costs are identical and normalized to zero for the sake of simplicity. Without loss of generality, we can associate the Western firm with the technologically advanced producer and normalize its marginal production costs to zero. The marginal production costs of the Eastern firm are denoted by \( c \geq 0 \).

There is a unit mass of consumers which is uniformly distributed over the unit interval. Consumers make a binary choice of purchasing one unit of the consumption good or nothing. They are identical with respect to their willingness to pay which we denote by \( A \). A consumer’s address is \( b \in [0,1] \). The two producers set mill prices \( p_\ell \) and consumers have to bear the shipping costs of \((b - x_\ell)^2\), which are quadratic in order to ensure existence of an equilibrium (see d’ Aspremont, Gabszewicz, and Thisse, 1979). Accordingly, the consumer price equals \( p(b, x_\ell) = p_\ell + (b - x_\ell)^2 \) for a consumer at address \( b \) who purchases the good from a producer located at \( x_\ell \).

By maximizing utility, consumers choose the supplier who offers the lower consumer price. To focus on the relevant aspects of the model, we impose two further assumptions. First, the willingness to pay \( (A) \) is sufficiently high to ensure full coverage of consumers in equilibrium: \( A > 5/4 + c/2 + c^2/36 \). Second, production cost differences are sufficiently small to guarantee positive demand for each of the two producers in equilibrium: \( c \leq 6 - \sqrt{27} \). See the formal analysis below for further details on these assumptions. Then, the relevant range of consumer demand depends on the address of the marginal consumer, which is determined by the indifference condition \( p(b_i, x_W) = p(b_i, x_E) \):

\[
b_i = \frac{x_W + x_E}{2} + \frac{p_W - p_E}{2(x_W - x_E)}.
\]

Consumer demand is given by \( d_W = b_i \) for the producer with address \( x_W \) and by \( d_E = 1 - b_i \)

\[5\text{In the borderline case of }x_W = x_E, \text{the model reduces to one with perfect price competition.}\]
for its competitor with address $x_E$. The corresponding profits of the two producers are

$$\pi_W = p_W b_i$$

$$\pi_E = (p_E - c)(1 - b_i).$$

Profit maximization entails two stages. The producers choose their location in the first stage and set prices subsequently. The maximization problem can be solved through backward induction. For given locations, the price reaction functions are

$$p_W = \frac{p_E}{2} + \frac{(x_E - x_W)(x_W + x_E)}{2}$$

$$p_E = \frac{p_W + c}{2} + \frac{(x_E - x_W)[2 - (x_W + x_E)]}{3},$$

according to (1)-(3). The two reaction functions confirm the well-known result that mill prices are strategic complements. By virtue of (4) and (5), sub-game-perfect equilibrium prices at stage two are given by

$$p^*_W(x_W, x_E) = \frac{c}{3} + \frac{(x_E - x_W)(2 + x_W + x_E)}{3}$$

$$p^*_E(x_W, x_E) = \frac{2c}{3} + \frac{(x_E - x_W)[4 - (x_W + x_E)]}{3}.$$ 

Substituting (6) and (7) in (1), we can express the marginal consumer’s address as a function of firm location:

$$b^*_i(x_w, x_E) = \frac{c}{6(x_E - x_W)} + \frac{2 + x_W + x_E}{6}.$$ 

Furthermore, substituting (6)-(8) in (2) and (3), profits can be expressed as a function of $x_W$ and $x_E$:

$$\pi^*_W(x_W, x_E) = \frac{1}{18(x_E - x_W)}\left\{ c + 2(x_E - x_W) + (x_E^2 - x_W^2) \right\}^2$$

$$\pi^*_E(x_W, x_E) = \frac{1}{18(x_E - x_W)}\left\{ -c + 4(x_E - x_W) - (x_E^2 - x_W^2) \right\}^2.$$ 

Solving for the profit-maximizing location choices and using superscript $n$ to refer to an equilibrium with independent producers (no merger), the following proposition can be established.
Proposition 1 Consider \( A > 5/4 + c/2 + c^2/36 \) and \( c \leq 6 - \sqrt{27} \). Then, the two firms locate at the boundaries of the unit interval (\( x^W_n = 0 \), \( x^E_n = 1 \)), the marginal consumer resides at address \( b^n_i = 1/2 + c/6 \), prices are given by \( p^n_W = c/3 + 1 \), \( p^n_E = 2c/3 + 1 \), and profits are given by \( \pi^n_W = (3 + c)^2/18 \), \( \pi^n_E = (3 - c)^2/18 \), respectively.

Proof. See Appendix.

Proposition 1 confirms the well-known result of maximum differentiation (in firm location) if transport costs are quadratic (see d’Aspremont, Gabszewicz, and Thisse, 1979). In addition, we see that an increase of production cost parameter \( c \) not only leads to a higher mill price of the Eastern producer, but also implies a higher mill price of the Western firm, as prices are strategic complements. However, the price increase of the Eastern firm is larger, so that the marginal consumer moves eastwards. As a consequence, the market share of the Western firm increases, while the market share of the Eastern firm declines.\(^6\)

In the following, we associate the Hotelling line with two integrated regions – thereby abstracting from any additional costs of shipping goods across regional borders (see Tharakan and Thisse, 2002). The Western region is of length \( r \) and the Eastern one of length \( 1 - r \). In this case, the trade pattern depends on the location of the regional border \( r \) relative to the address of the marginal consumer. If \( b^n_i > r \), the Western region exports the consumption good, while the Eastern region exports, if \( b^n_i < r \). In the case of identical production costs (\( c = 0 \)), it is the smaller region that exports due to lower transport costs for serving consumers at the common border (see Tharakan and Thisse, 2002). However, in the case of cost asymmetries the differential \( c > 0 \) matters as well (see Egger and Egger, 2007).

A final element we are interested in is the welfare level, which depends on the profit maximizing location and price choices of firms. Overall (world) welfare equals the sum of total profits \( \Pi^n \equiv \pi^n_W + \pi^n_E = 1 + c^2/9 \) and consumer surplus \( CS \), with the latter being

\(^6\)Note that \( p(b^n_i, 0) = p(b^n_i, 1) = 5/4 + c/2 + c^2/36 \) implies full coverage of consumers in the pre-merger equilibrium.
given by

\[ CS^n = \int_0^{b^n} [A - p^n_W - b^2]db + \int_{b^n}^1 [A - p^n_E - (1 - b)^2]db \]

\[ = A - 4/3 - c/3 + (1/2 - c/6)^2. \]  

(11)

Hence, overall welfare is

\[ V^n = A - 1/12 - c/2 + 5c^2/36. \]

(12)

It is intuitive that welfare declines in cost parameter \( c \), because a higher \( c \) can be associated with a less efficient production technology in the East.

To determine regional welfare levels, the ownership structure of firms is important. For simplicity, we assume that ownership of firms (and thus total profit income \( \Pi^n \)) is equally distributed among consumers.\(^7\) Then, profit income in the Western region is given by \( r\Pi^n \), while profit income in the Eastern region equals \( (1 - r)\Pi^n \). Noting further that regional consumer surplus is given by

\[ CS^n_W(r) = \begin{cases} [A - c/3 - 1]r - r^3/3 & \text{if } r \leq b_i^n \\ [A - c/3 - 1]r - r^3/3 + (r - 1/2 - c/6)^2 & \text{if } r > b_i^n \end{cases} \]

and

\[ CS^n_E(r) = \begin{cases} (A - c/3 - 1)(1 - r) + (1/2 - c/6)^2 - 1/3 + r^3/3 & \text{if } r \leq b_i^n \\ \int_r^1 [A - 2c/3 - 1 + r](1 - r) - 1/3 + r^3/3 & \text{if } r > b_i^n \end{cases} \]

we obtain

\[ V^n_W(r) = \begin{cases} [A - c/3 + c^2/9]r - r^3/3 & \text{if } r \leq b_i^n \\ [A - c/3 + c^2/9]r - r^3/3 + (r - 1/2 - c/6)^2 & \text{if } r > b_i^n \end{cases} \]

\[ (13) \]

\[ (14) \]

\[ (15) \]

\(^7\)Note that this assumption differs from the respective assumption in Tharakan and Thisse (2002), where firm ownership is region-specific. In the context of mergers, however, our approach is more convenient, because it does not require any further assumptions about the inter-regional distribution of merger gains. Similar assumptions regarding firm ownership can be found in the literature on international tax competition. See Haufler and Schjelderup (2000) for an example.
and

\[
V^n_E(r) = \begin{cases} 
(A - c/3 + c^2/9)(1 - r) + (1/2 - c/6)^2 - 1/3 + r^3/3 & \text{if } r \leq b^n_i \\
\int_r^1 [A - 2c/3 + c^2/9 + r](1 - r) - 1/3 + r^3/3 & \text{if } r > b^n_i 
\end{cases}
\]  

(16)

for the regional welfare levels in \( W \) and \( E \), respectively.

This completes our discussion of the pre-merger equilibrium. In the next section, we investigate how a merger (and thus joint profit maximization) affects location and price decisions. We are also interested to which extent a technology transfer and the use of the best-practice technology in both production plants influences these decisions. Furthermore, we compare welfare and the regional trade pattern in the pre- and post-merger equilibrium.

3 A merger between the two firms

To draw a comprehensive picture of the possible merger effects, we distinguish three alternative scenarios. In the first one, we assume that \( c = 0 \) holds both before and after the merger. This benchmark analysis allows us to investigate in detail the role played by changes in the location decision and price-setting behavior of firms for the variables of interest. In the second scenario, we allow for technology differences and consider asymmetric production costs which are the same before and after the merger takes place: \( c > 0 \). In the third scenario, we account for production cost differences in the pre-merger case, \( c > 0 \), and assume that the merger leads to a harmonization of production costs: \( c = 0 \). A comparison of scenarios 2 and 3 highlights the consequences of a technology transfer.

3.1 Costs are identical ex ante and ex post

If \( c = 0 \), firms do not differ in their production costs. They set identical prices \( (p^n_W = p^n_E = c + 1) \) and realize the same level of profits \( (\pi^n_W = \pi^n_E = 1/2) \) in the pre-merger equilibrium (see Proposition 1). The two producers share the market equally \( (b^n_i = 1/2) \),
so that it is the smaller region, i.e., region $W$ if $r < 1/2$ and region $E$ if $r > 1/2$, that exports the consumption good. This corresponds to the case of a long-run free trade equilibrium in Tharakan and Thisse (2001). Fig. 1 gives a graphical representation of the price-location schedules in the pre-merger equilibrium (dashed lines).

Figure 1: Price location schedules in the pre- and post-merger equilibrium if production costs do not differ across firms.

A merger of the two firms leads to a monopolization of the market and, therefore, to higher mill prices. To be more specific, by maximizing joint profits, the integrated firm increases prices until the most distant consumer that is served from a particular plant is indifferent between purchasing and not purchasing the consumption good. In addition, moving both production sites to the interior of the market reduces transport costs and allows for a further increase in the mill prices. If production costs are identical, the integrated firm will choose those locations for its two production facilities, which minimize overall transport cost expenditures. Altogether, joint profit-maximization leads to plant locations $x^m_W = 1/4$, $x^m_E = 3/4$ and prices $p^m_W = p^m_E = A - 1/16$ (with superscript $m$.
refering to a post-merger equilibrium variable). In this case, the marginal (most distant) consumer resides at address \( b^m_i = 1/2 \) and there is full coverage in equilibrium. The corresponding profits are given by \( \Pi^m = A - 1/16 \).

Comparing profits in the post-merger equilibrium, \( \Pi^m \), with total profits in the pre-merger situation, \( \Pi^n = 1 \), we can see that a merger leads to profit gains of \( \Delta \Pi = A - 17/16 > 0 \). Besides these positive profit effects, a merger also influences consumer surplus, with the respective change being given by

\[
\Delta CS = 17/16 - A + \int_0^{1/2} [b/2 - 1/16] \, db + \int_{1/2}^1 [7/16 - b/2] \, db
\]

\[
= 18/16 - A,
\]

(17)

which is negative if \( A > 5/4 + c/2 + c^2/36 \). Hence, a merger of the two firms exhibits two counteracting effects on overall (world) welfare. On the one hand, it raises profit income \( \Pi \), but on the other hand, it reduces consumer surplus \( CS \). For given aggregate consumer demand the profit gain dominates the consumer surplus loss, implying that overall welfare increases:

\[
\Delta V \equiv \Delta CS + \Delta \Pi = 1/16 > 0.
\]

(18)

This welfare increase is due to a decline in overall transport cost expenditures and equals \( \Delta T \equiv T^n - T^m \) in Fig. 1.

The increase in overall (world) welfare, however, does not mean that both regions can equally participate in the respective gains. To determine the regional welfare effects, let us first look at consumer surplus changes in the Western region. They are given by

\[
\Delta CS_W(r) = \begin{cases} 
(17/16 - A)r + (r/4) \left[ r - 1/4 \right] & \text{if } r \in [0, 1/2] \\
(24/16 - A)r - r^2/4 - 4/32 & \text{if } r \in (1/2, 1].
\end{cases}
\]

(19)

\[\text{8The consumer surplus in the pre-merger equilibrium is given by } A - 13/12, \text{ according to (11), while the consumer surplus in the post-merger equilibrium equals } CS^m = \int_0^{1/2} \left[ 1/16 - (b - 1/4)^2 \right] \, db + \int_{1/2}^1 \left[ 1/16 - (b - 3/4)^2 \right] \, db = 1/24. \text{ Noting } \Delta CS \equiv CS^m - CS^n, \text{ the second line in (17) follows immediately.}\]
Noting further that profit gains in $W$ equal $\Pi_W(r) = [4 - 17/16]r$, we find that welfare effects in the Western region are given by

$$\Delta V_W(r) = \begin{cases} 
  (r/4) [r - 1/4] & \text{if } r \in [0, 1/2] \\
  (7/16)r - r^2/4 - 2/16 & \text{if } r \in (1/2, 1].
\end{cases}$$

(20)

From (20), we can conclude that the Western region is worse off after the merger if $r < 1/4$, while it benefits from the merger if $r > 1/4$. This result is intuitive, when looking at Fig. 1. There, we can see that consumers on interval $[0, b_W)$ experience a transport cost increase after the merger, as the Western production facility moves eastwards. Hence, this group of individuals definitely loses. Furthermore, comparing total regional transport cost expenditures in the pre- and the post-merger equilibrium confirms our analytical result. Due to symmetry in the production costs, we can also conclude that the Eastern region is better off after the merger if $r < 3/4$, while it is worse off if $r > 3/4$.

A final issue we have to address is the impact of a merger on the trade structure. From Fig. 1 we see that the merger does not influence the position of the marginal consumer, i.e., $b_W^n = b_W^m = 1/2$. Hence, recollecting from above that the small region exports the consumption good in the pre-merger equilibrium, it is immediately clear that the direction of trade is unaffected by the merger, as long as $1/4 < r < 3/4$. In this case, the Western production plant remains located in the Western region and the Eastern production plant remains located in the Eastern region. If, however, $r < 1/4$ or $r > 3/4$, the smaller region loses its production facility and, thus, becomes an importer of the consumption good. In this case, a merger reverses the direction of trade.

Proposition 2 summarizes the most important results of the previous analysis.

**Proposition 2** If $c = 0$, a merger raises profits, reduces consumer surplus and increases overall (world) welfare. If $r < 1/4$ ($r > 3/4$), welfare declines in the small Western

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9From the first line of (20), it is immediately clear that $\Delta V_W(r)$ is negative if $r \in (0, 1/4)$, while $\Delta V_W(r)$ is positive if $r \in (1/4, 1/2]$. Furthermore, defining $B(r) \equiv (7/16)r - r^2/4 - 2/16$, with $B'(r) > 0$, $=, < 0 \iff 7/8 >, =, < r$ and noting $B(1/2) = 1/32 > 0$, $B(1) = 1/16 > 0$, we can conclude that $\Delta V_W(r) > 0$ for all $r \in (1/2, 1]$.
(Eastern) region and the direction of trade is reversed. On the contrary, if $1/4 < r < 3/4$ both regions benefit from a merger and the smaller region exports the consumption good in the pre- as well as the post-merger equilibrium.

**Proof.** Analysis in the text. ■

While the assumption of equal production costs provides an interesting benchmark for our analysis, it seems natural from the viewpoint of empirical stylized facts to consider production cost differences across firms as an alternative. Therefore, in a next step we investigate how the results in Proposition 2 change if firms differ in their technologies. For this purpose, we consider $c > 0$ in subsection 3.2.

### 3.2 Costs are different ex ante and ex post

If $c > 0$, the more productive Western producer sets a lower mill price, serves a larger share of consumers and earns higher profits in the pre-merger equilibrium: $p^n_W < p^n_E$, $b^n_i = 1/2 + c/6$ and $\pi^n_W > \pi^n_E$, according to Proposition 1. In addition, there may be exports of $W$ even if it is the larger region. To be more specific, $W$ exports the consumption good, as long as $r < b^n_i$, with $b^n_i > 1/2$ if $c > 0$. In line with Egger and Egger (2007), we can draw the following conclusion: The Western region exports the consumption good if its size disadvantage is not too pronounced relative to the production cost advantage of its local producer.

The analysis of the post-merger equilibrium becomes somewhat more complicated than the respective analysis in subsection 3.1. In particular, we can distinguish between three sources of profit gains if $c > 0$. First, for a given locational choice, the integrated producer can increase either mill price, because $p(b^n_i, 0) = p(b^n_i, 1) < A$ if $A > 5/4 + c/2 + c^2/36$. Second, by moving both production sites to the interior of the market, overall transport costs decline, so that mill prices can be further increased without reducing overall consumer demand. Third, in addition to these two types of profit gains, which are also present in the case of identical production costs, the integrated firm has an incentive to increase the market share of its low-cost Western production facility: $d_W > d_E \geq 0$. 

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Two subcases can be distinguished with respect to the size of \( d_E \). If the production cost disadvantage of the Eastern plant is sufficiently small \((0 < c < 3/4)\), the integrated firm will operate two production plants at locations \( x_W \in (1/4, 1/2) \) and \( x_E = (3/4, 1) \), respectively. In this case, we have \( d_E > 0 \). If, however, production cost differences are sizable \((3/4 \leq c \leq 6 - \sqrt{27})\), the integrated firm shuts down the Eastern production facility and serves all consumers from the center of the market to minimize overall transport costs: \( x_W = 1/2 \). This implies \( d_W = 1 \) and \( d_E = 0 \). We discuss these two subcases, separately.

**Case I:** \( 0 < c < 3/4 \)

If the integrated firm operates two production plants, it serves consumers on interval \([0, b_i]\), with \(0 < b_i < 1\), from its Western production facility and consumers on interval \((b_i, 1]\) from its Eastern production facility. In this case, profit-maximization is associated with location choices \( x_W = b_i/2, x_E = 1/2 + b_i/2 \), prices \( p_W = A - b_i^2/4, p_E = A - (1 - b_i)^2/4 \), and joint profits \( \Pi = A - c - 1/4 + b_i(c + 3/4) - 3b_i^2/4 \). Differentiating the latter expression, with respect to \( b_i \), we can conclude that \( b_i < 1 \) requires \( c < 3/4 \). In this case, the marginal consumer has address \( b_m^i = 1/2 + 2c/3 \), the two production plants are located at \( x_W^m = 1/4 + c/3, x_E^m = 3/4 + c/3 \), respectively, mill prices are given by \( p_W^m = A - (1/4 + c/3)^2 \), \( p_E^m = A - (1/4 - c/3)^2 \) and joint profits equal \( \Pi^m = A - 1/16 - c^2/2 + c^2/3 \). Fig. 2 displays the pre-merger and the post-merger equilibrium.

The monopolization of the market in the post-merger equilibrium leads to an increase in profit income, \( \Delta \Pi = A - 17/16 - c/2 + 2c^2/9 > 0 \), while the consumer surplus declines, \( \Delta CS = 18/16 - A + c/2 + 7c^2/36 < 0 \) (consider \( A > 5/4 + c/2 + c^2/36 \)). Summing up, we obtain\(^{11} \)

\[
\Delta V = 1/16 + 5c^2/12,
\]  

\(^{10}\)Similar to our analysis in subsection 3.2 we can conclude that if the integrated firm wants to serve consumers on some interval \([b_l, b_r]\) through production in its Western (Eastern) plant, it is always the best strategy to locate this plant in the center of the interval, in order to minimize transport costs.

\(^{11}\)See the Appendix for a detailed derivation of (21).
which is positive and strictly increasing in \( c \). With a higher cost differential, the integrated producer has an incentive to increase the market share of the Western plant, by moving both production facilities eastwards. (Formally, we have \( dx_W^m/dc = dx_E^m/dc = 1/3 \).) This reduces the social costs of a higher \( c \). Since an adjustment of firm location is not feasible in the pre-merger equilibrium (at least not as long as \( c \leq 6 - \sqrt{27} \)), it is intuitive that a higher cost differential \( c \) has a positive impact on \( \Delta V \).

![Price location schedules in the pre- and post-merger equilibrium if the production cost differential across firms is small.](image)

Figure 2: Price location schedules in the pre- and post-merger equilibrium if the production cost differential across firms is small.

With the overall welfare effects at hand, we can now turn to the regional implications of the merger. Since the formal derivation of the regional welfare effects is tedious, we have relegated it to the Appendix with the most important insights being summarized in the following lemma.

**Lemma 1** If \( c \in (0, 3/4) \) and \( r \in (0, 1) \), there exists a critical level \( \bar{r}_W \equiv [1/16 + c/6 - 2c^2/9]/(1/4 + c/3) \), such that \( \Delta V_W(r) >, =, < 0 \) if \( r >, =, < \bar{r}_W \). Welfare in the Eastern region definitely increases if \( c \geq -3/8 + \sqrt{27}/8 \). If, however, \( c < -3/8 + \sqrt{27}/8 \), then there
exists a critical level \( \bar{r}_E \equiv \left[ 2c^2/9 - c/6 + 3/16 \right] / \left( 1/4 - c/3 \right) \), such that \( \Delta V_E(r) >, =, < 0 \) if \( \bar{r}_E >, =, < r \).

**Proof.** See Appendix. ■

For an intuition of the regional welfare effects in Lemma 1, it is meaningful to contrast these results with the respective findings in subsection 3.1. In the case of identical production cost \((c = 0)\), the Western region experiences a welfare loss (gain) in the form of higher (lower) transport cost expenditures if \( r < (>)1/4 \). Furthermore, we know from (21) that a higher \( c \) tends to increase the social surplus, due to an adjustment in firm location: \( dx_{mW}^w/dc > 0, dx_{mE}^e/dc > 0 \). This effect tends to lower the critical level of \( r \).

On the other hand, an increase in \( x_{mW}^w \) leads to higher transport costs for consumers on interval \([0, x_{mW}^w]\), which counteracts the first effect and tends to shift the critical level of \( r \) eastwards. If \( c > 3/8 \), it is the first effect that dominates, implying \( \bar{r}_W < 1/4 \). In contrast, if \( c < 3/8 \) the second effect is stronger, so that \( \bar{r}_W > 1/4 \).

Things are different in the Eastern region, where \( 3/4 \) gives the critical level of \( r \) if \( c = 0 \) (see Proposition 2). In the Eastern region, the two identified effects of an increase in \( c \) go in the same direction. For sufficiently high levels of \( c \) (i.e., if \( c > -3/8 + \sqrt{27}/8 \)), this implies that the Eastern region will always benefit from a merger of the two firms. In contrast, if \( c < -3/8 + \sqrt{27}/8 \), there exits a critical level \( \bar{r}_E \in (3/4, 1) \), such that the Eastern region is better (worse) off after the merger, if \( \bar{r}_E > (<)r \).

A final element we have to determine is the role of a merger for the regional trade pattern. Similar to subsection 3.1, we can note that \( r < x_{mW}^w \) implies that the Western region exports the consumption good in the pre-merger equilibrium, while it loses its local production facility and, therefore, imports the consumption good in the post-merger equilibrium. In analogy, \( r > x_{mE}^e \) implies that the Eastern region loses its local production facility and imports the consumption good in the post-merger equilibrium (although it was an exporter of the consumption good in the pre-merger equilibrium). However, in contrast to subsection 3.1, where the address of the marginal consumer was not influenced by a merger of the two firms, we obtain \( b_i^w < b_i^m \) if \( c > 0 \). Hence, there is a third \( r \)-domain, where a merger reverses the direction of trade. If \( r \in (b_i^w, b_i^m) \), the Western region imports
the commodity in the pre-merger equilibrium, while it becomes an exporter in the post-
merger equilibrium. This result points to a non-trivial interplay of size and production
cost differences, because the direction of trade may not only be reversed if regions differ
substantially in their size but also if the size difference is rather small.

In a final thought experiment, we can sum up the different ranges, in which a trade
reversal occurs, and obtain $R = 1/2 + c/2$ if $c \in (0, 3/4)$. This implies that the likelihood
of a trade reversal increases with the cost differential $c$, if region size is randomly drawn
from the unit interval. This completes our formal analysis of case I with a relatively small
cost differential $c \in (0, 3/4)$. In a next step, we investigate case II, in which the cost
differential is more pronounced: $c \in [3/4, 6 - \sqrt{27}]$.

**Case II: $3/4 \leq c \leq 6 - \sqrt{27}$**

If $c \geq 3/4$, operating the high-cost Eastern production facility becomes unattractive for
the integrated firm.\(^{12}\) In this case, profits are maximized by choosing location $x^m_W = 1/2$
and setting a mill price $p^m_W = A - 1/4$. With the Western plant serving all consumers
($d_W = 1$), this implies that profits equal $\Pi^m = A - 1/4$ in the post-merger equilibrium.
Fig. 3 illustrates the pre- and the post-merger equilibrium for $c \in [3/4, 6 - \sqrt{27}]$.

Profits gains are given by $\Delta \Pi^m = A - 5/4 - c^2/9$ and the consumer surplus change
$\Delta CS = 5/4 + c/2 - c^2/36$ is negative due to our assumption about $A$. Summing up,
overall welfare changes can be expressed in the following way\(^ {13}\)

$$\Delta V = c/2 - 5c^2/36.$$  \hfill (22)

Noting that $c \leq 6 - \sqrt{27}$, we can conclude that a merger raises welfare, i.e., $\Delta V > 0$, with
the respective welfare gain being increasing in the cost differential $c$. This confirms the
respective result of case I. Furthermore, the regional welfare effects can be summarized
as follows.

---

\(^{12}\)Note that $\lim_{c \rightarrow 3/4} b^i_W = 1$ holds under case I.

\(^{13}\)See the Appendix for a detailed derivation of (22).
Lemma 2 If $c \in \left[3/4, 6 - \sqrt{27}\right]$ and $r \in (0, 1)$, there exists a critical $\bar{r}_W = 1/2 - 2c/3 + 2c^2/9$, such that $\Delta V_W(r) > = < 0$ if $r > = < \bar{r}_W$. Welfare in the Eastern region increases, i.e., $\Delta V_E(r) > 0$, for any $r$.

Proof. See Appendix.

The results in Lemma 2 confirm our previous insight that the Western region is better off after the merger if it is sufficiently large ($r > \bar{r}_W$, with the critical region size, $\bar{r}_W$, being smaller than $1/4$ if $c \geq 3/4$), while the Eastern region always benefits if the cost differential is sizable (with $c \geq 3/4$ being sufficient).

![Figure 3: Price location schedules in the pre- and post-merger equilibrium if the production cost differential across firms is sizable.](image)

A final issue to be addressed is the impact of a merger on the trade pattern. In Fig. 3, we see that a merger reverses the direction of trade if $r < 1/2$. In this case, the smaller Western region exports in the pre-merger equilibrium, while it becomes an importer after the merger, because the consumption good is produced in the larger Western region in the post-merger equilibrium. Furthermore, if $r > b_i^n$, region $E$ exports in the pre-merger equi-
librium, while it imports the consumption good in the post-merger equilibrium. Finally, the merger leaves the direction of trade unaffected, if \( r \in (1/2, b^n) \). Hence, in contrast to case I, there are only two intervals, where a merger changes the direction of trade, if \( c \in [3/4, 6 - \sqrt{27}] \).

Similar to case I, we can sum up the parameter ranges over which a merger reverses the direction of trade and obtain \( R = 1 - c/6 \). Hence, an increase of cost differential \( c \) reduces the likelihood of a trade reversal if region size is randomly drawn from the unit interval. Together with the insights of case I, this implies that the likelihood of a trade reversal reaches a maximum at \( c = 3/4 \).

With the formal analysis of cases I and II at hand, we can now summarize the main effects of a merger on welfare and the trade pattern.

**Proposition 3** If \( c > 0 \) a merger raises profits and reduces consumer surplus. Overall welfare goes up, with the respective gain being increasing in the cost differential \( c \). The Western region benefits only, if it is sufficiently large, while the Eastern region always benefits, if the cost differential \( c \) is not too small, i.e., if \( c > -3/8 + \sqrt{27}/8 \). Otherwise (if \( c < -3/8 + \sqrt{27}/8 \), the Eastern region may be worse off after the merger, if it is sufficiently small. Regarding the trade pattern effects, we find that the likelihood of a trade reversal is always higher in the case of cost asymmetry \((c > 0)\) than in the case of identical production costs \((c = 0)\) and that it reaches a maximum at \( c = 3/4 \).

**Proof.** Proposition 3 follows from the anlysis above.

### 3.3 Costs are different ex ante but identical ex post

In the previous two subsections, the focus of our analysis was on the role of joint profit maximization for plant location, prices, welfare and the trade pattern. Thereby, we assumed that production costs were not influenced by the merger decision. In this subsection, we allow for a technology transfer and assume that both plants use the best-practice technology in the post-merger equilibrium. Of course, a technology transfer is only relevant if production costs differ *ex ante*. Then, prices are given by \( p^n_W = 1 + c/3 \)
\( p_E^n = 1 + 2c/3 \) and the marginal consumer has address \( b_i^n = 1/2 + c/6 \) in the pre-merger equilibrium. Profits of the Western and Eastern firm are given by \( \pi_W^n = (3 + c)^2/18 \) and \( \pi_E^n = (3 - c)^2/18 \), respectively, according to Proposition 1.

![Figure 4: Price location schedules in the pre- and post-merger equilibrium if a technology transfer leads to a harmonization of production costs.](image)

In the post-merger equilibrium, production costs are identical (due to the technology transfer) and profit maximization implies plant locations \( x_W^m = 1/4, \ x_E^m = 3/4 \) and prices \( p_W^m = p_E^m = A - 1/16 \), according to our analysis in subsection 3.1. Both plants serve half of the market, \( b_i^n = 1/2 \), and profits are given by \( \Pi^m = A - 1/16 \). Fig. 4 illustrates this scenario. Turning to the welfare effects, we can first note that profit gains, \( \Delta \Pi = A - 17/16 - c^2/9 \) depend negatively on cost parameter \( c \). On the one hand, a higher \( c \) reduces the intensity of price competition in the pre-merger equilibrium, thereby leading to higher total profits: \( d\Pi^n/dc = 2c/9 > 0 \). On the other hand, a higher \( c \) has no impact on profits in the post-merger equilibrium, as both production plants use the best-practice technology (which is characterized by zero production costs). Furthermore, the consumer
surplus change is given by \( \Delta CS = [18/16 + c/2 - c^2/36 - A] \), which is negative due to our assumption about \( A \). Summing up, we obtain\(^\text{14}\)

\[
\Delta V = 1/16 + c/2 - 5c^2/36 > 0.
\]

Similar to the analysis in subsection 3.2, a higher cost differential \( c \) raises the merger-induced welfare gain. However, \( \Delta V \) in (23) is larger than the respective values in (21) and (22). As compared to case I, if a merger leads to a technology transfer, there are additional efficiency gains from merging production sites, as the inferior Eastern production technology is replaced by the superior Western technology. In case II, the whole market was served by a single plant (using the Western technology), so that two-plant production exhibits a welfare gain due to a decline in overall transport cost expenditures. Let us now turn to the regional welfare effects, with the main insights being summarized in the following lemma.

**Lemma 3** Consider \( r \in (0, 1) \). If the merger leads to a technology transfer and the use of the best-practice technology in both production plants, the following regional welfare effects can be derived. First, if \( c \geq 6/4 - \sqrt{27}/4 \), then \( \Delta V_W(r) > 0 \) for any \( r \in (0, 1) \). In contrast, if \( c < 6/4 - \sqrt{27}/4 \), then there exists a critical \( r^2_W \equiv 1/4 - 4c/3 + 4c^2/9 \), such that \( \Delta V_W(r) >, =, < 0 \) if \( r >, =, < r^2_W \). Second, welfare in the Eastern region unambiguously increases, if \( c \geq 3(1 - \sqrt{15/16}) \), while \( c < 3(1 - \sqrt{15/16}) \) implies that there exists a critical \( r^2_E \equiv 3/4 + 8c/3 - 4c^2/9 \), such that \( \Delta V_E(r) >, =, < 0 \) if \( r^2_E >, =, < r \).

**Proof.** See Appendix. ■

Lemma 3 confirms our previous insight that a technology transfer provides an additional source of welfare gain. As the positive effect of a technology transfer increases with the cost differential \( c \), it is intuitive that even very small regions can benefit from a merger if \( c \) is sufficiently large.

Let us now turn to the trade pattern effects. Similar to subsection 3.1, we can identify \( r = 1/4 \) and \( r = 3/4 \) as two critical levels of \( r \) for a trade-reversing effect of a merger. If

\(^{14}\)See Appendix for a detailed derivation of (23).
If $r < 1/4$, the Western region exports the consumption good in the pre-merger equilibrium, while it loses its local production facility and, therefore, becomes an importer in the post-merger equilibrium. In analogy, if $r > 3/4$ the Eastern economy loses its local production facility and becomes an importer of the consumption good in the post-merger equilibrium (although it was an exporter in the pre-merger equilibrium). The trade reversal in these two cases arises due to an adjustment in firm location. However, similar to the analysis in subsection 3.2 (case I), there is a third parameter range, where a trade reversal occurs. If $r \in (1/2, b^n)$, the Western region exports the consumption good in the pre-merger equilibrium, while it imports the consumption good in the post-merger equilibrium. In this case, the trade reversal occurs due to a shift in the address of the marginal consumer.

We can now sum up the different ranges, where a trade reversal occurs and obtain $R = 1/2 + c/6$. Interpreting $R$ as the likelihood of a trade reversal after the merger if country size $r$ is randomly drawn from the unit interval, we can conclude that this likelihood increases with the ex ante cost differential $c$ and is smaller than the respective values in subsection 3.2. Hence, all other things equal, a technology transfer reduces the likelihood of a trade reversal after the merger. This completes our formal analysis of subsection 3.3, with the main insights on welfare and trade structure effects being summarized in the following proposition.

**Proposition 4** A technology transfer reinforces the positive welfare effects of a merger. Indeed, both regions are, irrespective of their country size, better off after the merger, if the ex ante cost differential and thus the gains from the technology transfer are sufficiently high. The likelihood of a trade reversal is reduced as compared to a scenario with asymmetric production costs and no technology transfer.

**Proof.** Proposition 4 follows from the analysis above. ■
4 Summary and concluding remarks

This paper uses a spatial model à la Hotelling to shed new light on the consequences of an inter-regional merger on welfare and the trade pattern. Starting point of the analysis is the long-run free trade equilibrium in Tharakan and Thisse (2002), with two asymmetrically sized regions, quadratic transport costs, and two firms located at the Western and Eastern end of the Hotelling line, respectively. In this setting, we show that joint profit maximization after the merger not only leads to an increase in mill prices but also to an adjustment in the location of production plants, which are moved to the interior of the market in order to reduce overall transport cost expenditures. In addition, an intra-firm technology transfer gives rise to an additional impact if firms \textit{ex ante} differ in their production costs. By separating these channels of influence, the analysis provides a detailed picture of the possible merger effects on the variables of interest.

With respect to the welfare effects, the main insights of our analysis can be summarized as follows. A merger raises profit income and reduces consumer surplus. Overall welfare unambiguously goes up, with the welfare gain being increasing in the \textit{ex ante} cost differential across firms. There are interesting regional implications, as well. Contrasting the results in this paper with the findings in Tharakan and Thisse (2002), we can formulate the following conclusion. While in the benchmark scenario with identical production costs, a movement from autarky to a long-run free trade equilibrium lowers welfare in the large region and, depending on the magnitude of the size difference, may render the small region better or worse off, a merger unambiguously increases welfare in the large region but lowers welfare in the small region if the size difference is sufficiently pronounced. With \textit{ex ante} production cost differences and a technology transfer after the merger, there are additional positive welfare effects, so that both regions can benefit from a merger, irrespective of the prevailing size differences.

Finally, our analysis also points to the possibility of a trade reversal after the merger. Such a trade reversal may either arise if the smaller region loses the local production facility after the adjustment in plant location or it may be triggered by a change in
the address of the marginal consumer, who is indifferent between purchasing from the two producers. Neither of these two explanations for a trade reversing effect can be discussed in a traditional trade model, in which countries are treated as points. In this respect, the paper contributes to a more general insight from studies that address the consequences of economic integration in a spatial framework. It is important to account for the geographical extension of countries, because globalization does not only influence the allocation of economic activity across borders but also within regions. For that reason, intra-regional effects require further attention in future research in order to obtain a deeper understanding and a more comprehensive picture about how globalization changes our lives.

Appendix

Proof of Proposition 1

In a first step, we show that \( c \leq 1 \) rules out an interior equilibrium with \( x_W \in (0, 1) \), \( x_E \in (0, 1) \) and \( b_i \in (0, 1) \). For this purpose, we hypothesize \( b_i \in (0, 1) \) and differentiate \( \pi_\ell(x_W, x_E) \) w.r.t. \( x_\ell \). This gives:

\[
\frac{\partial \pi_W(x_W, x_E)}{\partial x_W} = \frac{\pi_W(x_W, x_E)}{(x_E - x_W)} \left[ \frac{c}{x_E - x_W} - 2 - 3x_W + x_E \right] \quad (24)
\]

\[
\frac{\partial \pi_E(x_W, x_E)}{\partial x_E} = \frac{4[2 - x_E]\pi_E(x_W, x_E)}{-c + (4 - x_E - x_W)(x_E - x_W)} \left[ \frac{-c}{x_E - x_W} - 4 + 3x_E - x_W \right] \quad (25)
\]

According to (24) and (25), we can conclude that, for any \( x_W \leq x_E \), \( \partial \pi_W(\cdot)/\partial x_W < 0 \) and \( \partial \pi_E(\cdot)/\partial x_E > 0 \) if \( c = 0 \). This confirms the well-known result that two producers maximize the distance between their production sites in a linear model with quadratic transport costs and identical production costs (see d’Aspremont, Gabszewicz, and Thisse, 1979).
But what happens if production costs differ? To answer this question, note that \( \partial \pi_E(\cdot)/\partial x_E > 0 \) if \( c > 0 \). This implies \( x_E = 1 \). Furthermore, let us hypothesize that there exists a \( \tilde{x}_W \in (0, 1) \) that fulfills \( \partial \pi_W(x_W, 1)/\partial x_W = 0 \). From the second line of (24), we see that \( \partial \pi_W(x_W, 1)/\partial x_W = 0 \) requires \( c/(1 - x_W) - 1 - 3x_W = 0 \) and thus

\[
\tilde{x}_W^{1/2} = \frac{1}{3} \pm \sqrt{\frac{4}{9} - \frac{c}{3}}.
\]  

(26)

Noting further that

\[
\partial^2 \pi(\tilde{x}_W, 1)/\partial x_W^2 >, =, < 0 \iff \tilde{x}_W >, =, < 1/3,
\]  

(27)

follows from (24), it is obvious that \( \tilde{x}_W \leq 1/3 \) is required for a profit maximizing location decision on interval \((0, 1)\) – and \( b_i(\tilde{x}_W, 1) \in (0, 1) \). From (26), we can therefore conclude that \( c \leq 1 \) rules out an equilibrium with \( x_W \in (0, 1), x_E = 1 \) and \( b_i \in (0, 1) \).

So far, we have assumed that the equilibrium is characterized by positive demand of both producers. In principle, however, it may be attractive for the technologically advanced \( W \)-producer to deviate from \( x_W = 0 \) in order to serve all consumers. Profit-maximizing prices in this case are determined by \( p_W^D = \min[A - (x_W^D)^2, c - (1 - x_W^D)^2] \), with \( x_W^D \) being the optimal location if the Western producer serves the whole market. This location is determined by the following condition:

\[
x_W^D = \begin{cases} 
(A - c + 1)/2 & \text{if } A < c + 1 \\
1 & \text{if } A \geq c + 1
\end{cases}
\]  

(28)

Due to \( A > 5/4 + c/2 + c^2/36 \), we have \( x_W^D = 1 \), implying \( \pi_W^D = c \). Comparing \( \pi_W^D \) and \( \pi_W^n \), we can therefore conclude that deviation from \( x_W = 0 \) is unattractive if \((3 + c)^2/18 \geq c \) or, equivalently, if \( 9 - 12c + c^2 \geq 0 \). Together with our previous insights, we can therefore conclude that assumptions \( A > 5/4 + c/2 + c^2/36 \) and \( c \leq 6 - \sqrt{27} \) are sufficient for an equilibrium with \( x_W = 0, x_E = 1 \) and \( b_i \in (0, 1) \). Substituting the profit-maximizing location choices \((x_W = 0 \text{ and } x_E = 1)\) into (6)-(10) completes the proof of proposition 1.

QED.

\[^{15}\text{Straightforward calculations give } \partial \pi_W^2(\tilde{x}_W, 1)/\partial x_W^2 = c/(1 - x_W)^2 - 3. \text{ Substituting } c = (1 - \tilde{x}_W)(1 + 3\tilde{x}_W), \text{ according to (24), we further obtain } \partial \pi_W^2(\tilde{x}_W, 1)/\partial x_W^2 = 2(3\tilde{x}_W - 1)/(1 - \tilde{x}_W).\]
Consider $c \in (0, 3/4)$. According to Proposition 1 and our calculations in Subsection 3.2, a merger leads to a profit gain that equals

$$\Delta \Pi = \Pi^m - \Pi^n = A - 17/16 - c/2 + 2c^2/9.$$ (29)

The change in overall consumer surplus is given by

$$\Delta CS = \int_0^{b_i^n} [A - p_W^m - \left(b - x_W^m\right)^2] db + \int_{b_i^n}^1 [A - p_W^n - \left(b - x_W^n\right)^2] db$$

$$- \int_0^{b_i^n} [A - p_W^n - \left(b - x_W^n\right)^2] db - \int_{b_i^n}^1 [A - p_E^n - \left(b - x_E^n\right)^2] db$$

Noting $x_W^n = 0$, $x_E^n = 1$ and accounting for $b_i^n < b_i^m$, this simplifies to

$$\Delta CS = \int_0^{b_i^n} [p_W^m - p_W^n + 2x_W^m b - (x_W^n)^2] db$$

$$+ \int_{b_i^n}^1 [p_E^m - p_E^n + (1 - (x_E^m)^2 - 2(1 - x_E^n)b)] db + \int_{b_i^n}^1 [p_E^n - p_W^n + 1 - 2b] db.$$

Substituting $p_W^m - p_W^n = p_E^m - p_E^n = 17/16 - A + c/2 + c^2/9$, $p_E^m - p_W^m = c/3$, $b_i^m - b_i^n = c/2$, $x_W^m = b_i^m/2$ and $x_E^m = 1/2 + b_i^m/2$ further implies

$$\Delta CS = 17/16 - A + c/2 + 5c^2/18 + 2b_i^m/4 + (1/4)(1 - (b_i^m)^2) - b_i^n(1 - b_i^n),$$

which, accounting for $b_i^m = 1/2 + 2c/3$ and $b_i^n = 1/2 + c/6$, finally gives

$$\Delta CS = 18/16 - A + c/2 + 7c^2/36.$$ (30)

Then, noting $\Delta V = \Delta \Pi + \Delta CS$, (21) follows from (29) and (30). QED.

**Proof of Lemma 1**

Consider $c \in (0, 3/4)$ and $r \in (0, 1)$. Then, profit gains in the Western region (with size $r$) are given by $\Delta \Pi_W(r) = [A - 17/16 - c/2 + 2c^2/9] r$, according to (29). Furthermore, defining

$$B_1(r) \equiv \int_0^r [p_W^m - p_W^n + 2x_W^m b - (x_W^m)^2] db, \quad B_2(r) \equiv \int_{b_i^n}^r [p_E^m - p_W^n + 1 - 2b] db,$$

$$B_3(r) \equiv \int_{b_i^n}^r [p_E^m - p_E^n + (1 - (x_E^m)^2 - 2b(1 - x_E^n))] db,$$
consumer surplus changes in the Western region can be written in the following way

\[ \Delta CS_W(r) = \begin{cases} \quad B_1(r) & \text{if } r \in (0, b^n_i] \\ \quad B_1(r) + B_2(r) & \text{if } r \in (b^n_i, b^m_i] \\ \quad B_1(b^m_i) + B_2(b^m_i) + B_3(r) & \text{if } r \in (b^m_i, 1) \end{cases} \]

This expression can be simplified to

\[ \Delta CS_W(r) = \begin{cases} \quad [1 - A + c/3 + r(1/4 + c/3)]r & \text{if } r \in (0, b^n_i] \\ \quad [1 - A + c/3 + r(1/4 + c/3)]r - [r - (1/2 + c/6)]^2 & \text{if } r \in (b^n_i, b^m_i] \\ \quad [3/2 - A - r(1/4 - c/3)]r + c/6 + 7c^2/36 - 1/8 & \text{if } r \in (b^m_i, 1) \end{cases} \]

Noting that regional welfare is given by \( \Delta V_W(r) = \Delta \pi_W(r) + \Delta CS_W(r) \), we further obtain

\[ \Delta V_W(r) = \begin{cases} \quad [-1/16 - c/6 + 2c^2/9 + r(1/4 + c/3)]r & \text{if } r \in (0, b^n_i] \\ \quad [-1/16 - c/6 + 2c^2/9 + r(1/4 + c/3)]r - [r - (1/2 + c/6)]^2 & \text{if } r \in (b^n_i, b^m_i] \\ \quad [7/16 - c/2 + 2c^2/9 - r(1/4 - c/3)]r + c/6 + 7c^2/36 - 1/8 & \text{if } r \in (b^m_i, 1) \end{cases} \]

(31)

To determine the role played by \( r \) for the sign of \( \Delta V_W(r) \), let us separately consider the different parameter domains in (31). Accounting for \( c \leq 3/4 \) and using \( \bar{r}_W \equiv [1/16 + c/6 - 2c^2/9]/(1/4 + c/3) \), it follows from the first line of (31) that \( \Delta V_W(r) < 0 \) if \( r \in (0, \bar{r}_W) \), while \( \Delta V_W(r) > 0 \) if \( r \in (\bar{r}_W, b^n_i] \). In a next step, we can define

\[ D(r) \equiv [-1/16 - c/6 + 2c^2/9 + r(1/4 + c/3)]r - [r - (1/2 + c/6)]^2, \]

with \( \Delta V_W(r) = D(r) \) if \( r \in (b^n_i, b^m_i] \). Noting that\(^{16} \) \( D'(r) = 15/16 + c/6 + 2c^2/9 - r(3/2 - 2c/3) > 0 \) holds for all \( r \in [b^n_i, b^m_i] \), \( D(b^n_i) = [1/16 + c/24 + 5c^2/18](1/2 + 2c/3) > 0 \) assures a positive value of \( \Delta V_W(r) \) for any \( r \in (b^n_i, b^m_i] \). In a final step, we can define

\[ \bar{D}(r) \equiv [7/16 - c/2 + 2c^2/9 - r(1/4 - c/3)]r + c/6 + 7c^2/36 - 1/8, \]

\(^{16}\)Note that \( D'(r) > 0 \) holds for any \( r \in [b^n_i, b^m_i] \) if \( D'(b^m_i) = 3/16 - c/2 + 2c^2/3 > 0 \). Furthermore, \( D'(b^m_i) \) reaches a minimum at \( c = 3/8 \). Hence, we can conclude that \( D'(b^m_i) > 0 \) holds for any \( c \in [0, 3/4] \), if \( D'(b^m_i)|_c=3/8 > 0 \). Finally, evaluating \( D'(b^m_i) \) at \( c = 3/8 \) gives \( 3/32 > 0 \).
with $\bar{D}(r) = \Delta V_W(r)$ if $r \in (\bar{b}_t^m, 1)$. Differentiating $\bar{D}(r)$ gives $\bar{D}'(r) = 7/16 - c/2 + 2c^2/9 - r(1/2 - 2c/3)$, which is positive if $r = b_t^m$. Furthermore, $\bar{D}'(1) >, =, < 0 \Leftrightarrow c >, =, < -3/8 + \sqrt{27}/8$. In any case, accounting for $\bar{D}'(b_t^m) = 1/32 + c/8 + c^2/12 + 8c^3/27 > 0$ and $\bar{D}(1) = 1/16 + 9c^2/36 > 0$, it follows that $\Delta V_W(r)$ is positive for any $r \in (b_t^m, 1]$. Putting together, we can therefore conclude that $\Delta V_W(r) >, =, < 0$ if $r >, =, < \bar{r}_W$.

Let us now turn to the Eastern region. First, from the analysis above, we know that $\Delta V_W(r) < 0$ if $r \in (0, \bar{r}_W]$. Due to $\Delta V = \Delta V_W(r) + \Delta V_E(r) > 0$ (see (21)), this implies $\Delta V_E(r) > 0$ for any $r \in (0, \bar{r}]$. Second, noting that the expression in the first line of (31) is strictly increasing for all $r > \bar{r}_W$ and that the expression in the second line is strictly increasing in $r$ for any $r \in [b_t^m, b_t^m]$ (see our discussion above), we can safely conclude that $\Delta V_E(r)$ is positive for any $r \in [\bar{r}_W, b_t^m)$, if $\Delta V_E(b_t^m) > 0$. Third, let us hypothesize $\Delta V_E(b_t^m) > 0$ for the moment. Then, we can focus on interval $r \in [b_t^m, 1]$ in the subsequent analysis.

Profit gains of the Eastern region are given by $\Delta \Pi_E(r) = [A - 17/16 - c/2 + 2c^2/9](1-r)$ and the consumer surplus change amounts to

$$\Delta CS_E(r) = \int_r^1 [p_E^n - p_E^n + (1-b)^2 - (b - x_E^n)^2]db = [20/16 - A + c/3 - r(1/4 - c/3)](1-r),$$

if $r \in [b_t^m, 1]$. Summing up we obtain

$$\Delta V_E(r) = [3/16 - c/6 + 2c^2/9 - r(1/4 - c/3)](1-r),$$

(32)

for $r \in [b_t^m, 1]$. We can now define $F(r) \equiv 3/16 - c/6 + 2c^2/9 - r(1/4 - c/3)$, such that $rF = \Delta V_E(r)$, according to (32). Then, noting $F'(r) < 0$ and $F(3/4) = c/12 + c^2/9 > 0$, we can conclude that $\Delta V_E(r) > 0$ for any $r \in [b_t^m, 3/4]$. Together with the insights for interval $(0, b_t^m)$, this implies $\Delta V_E(r) > 0$ for any $r \in (0, 3/4]$. Evaluating $F(r)$ at $r = 1$, we further obtain $F(1) = -1/16 + c/6 + 2c^2/9$. Hence, we can conclude that $\Delta V_E(r) > 0$ for any $r \in (0, 1)$ if $c \geq -3/8 + \sqrt{27}/8$. Otherwise (if $c < -3/8 + \sqrt{27}/8$), there exists a critical $\bar{r}_E \equiv [3/16 - c/6 + 2c^2/9]/(1/4 - c/3)$, such that $\Delta V_E(r) >, =, < 0$ if $\bar{r}_E >, =, < 0$. This completes the proof of Lemma 1. QED.
Derivation of equ. (22)

Consider $c \in [3/4, 6 - \sqrt{27}]$. Then, using the results from Proposition 1 and insights from the analysis in subsection 3.2, we obtain

$$\Delta \Pi = \Pi^m - \Pi^n = A - 5/4 - c^2/9.$$  \hfill (33)

Furthermore, the change in consumer surplus is given by

$$\Delta CS = \int_0^{b^n} [p^n_W - p^n_W + b - 1/4]db + \int_{b^n}^{1} [p^n_E - p^n_W + 3/4 - b]db$$

$$= 5/4 + c/2 - c^2/36.$$  \hfill (34)

Summing up, gives (22). QED.

Proof of Lemma 2

Consider $c \in [3/4, 6 - \sqrt{27}]$ and $r \in (0, 1)$. Then, profit gains in the Western region (with size $r$) are given by $\Delta \Pi_W(r) = [A - 5/4 - c^2/9]r$. Furthermore, the consumer surplus change in $W$ is determined by

$$\Delta CS_W(r) = \begin{cases} 
\int_0^{b^n} [p^n_W - p^n_W + b - 1/4]db & \text{if } r \in (0, b^n] \\
\int_0^{b^n} [p^n_W - p^n_W + b - 1/4]db + \int_{b^n}^{1} [p^n_E - p^n_W + 3/4 - b]db & \text{if } r \in (b^n, 1) 
\end{cases}.$$  

Substituting $p^n_W - p^n_W = 5/4 + c/3 - A$, $p^n_E - p^n_W = 5/4 + 2c/3 - A$ and $b^n = 1/2 + c/6$ this gives

$$\Delta CS_W(r) = \begin{cases} 
[1 - A + c/3 + r/2]r & \text{if } r \in (0, b^n] \\
[2 + 2c/3 - A - r/2]r - (1/2 + c/6)^2 & \text{if } r \in (b^n, 1). 
\end{cases}$$

Taking into account that $\Delta V_W(r) = \Delta \Pi_W(r) + \Delta CS_W(r)$, welfare changes in $W$ are given by

$$\Delta V_W(r) = \begin{cases} 
[-1/4 + c/3 - c^2/9 + r/2]r & \text{if } r \in (0, b^n] \\
[3/4 + 2c/3 - c^2/9 - r/2]r - (1/2 + c/6)^2 & \text{if } r \in (b^n, 1). 
\end{cases}$$  \hfill (35)
To determine the sign of $\Delta V_W(r)$, note first that $\Delta V_W(r) > 0$ holds for any $r \in (b_i n^n, 1)$. Then, $\Delta V_W(b_i^n) = (5c/12 - c^2/9)(1/2 + c/6) > 0$, we find that $\Delta V_W(r) > 0$ for any $r \in (b_i^n, 1)$, according to the second line of (35). Noting further that $-1/4 + c/3 - c^2/9 < 0$ holds for any $c \leq 6 - \sqrt{27}$, it follows from the first line of (35) that there exists a critical $\bar{r}_W^1 = 1/2 - 2c/3 + 2c^2/9$, such that $\Delta V_W(r) > = 0$ if $r > = \bar{r}_W^1$.

Let us now turn to the welfare effects in the Eastern region. Profit changes in $E$ are given by $\Delta \Pi_E(r) = [A - 5/4 - c^2/9](1 - r)$. Furthermore, the consumer surplus change is determined by

$$\Delta CS_E(r) = \begin{cases} \int_r^{b_i^n} [p_{E}^n - p_{W}^n - 1/4 + b]db + \int_{b_i^n}^{1} [p_{E}^n - p_{W}^n + 3/4 - b]db & \text{if } r \in (0, b_i^n] \\ \int_r^{b_i^n} [p_{E}^n - p_{W}^n + 3/4 - b]db & \text{if } r \in (b_i^n, 1) \end{cases}$$

Substituting $p_{W}^n - p_{W}^m = 5/4 + c/3 - A$, $p_{E}^n - p_{W}^m = 5/4 + 2c/3 - A$ and $b_i^n = 1/2 + c/6$, gives

$$\Delta CS_E(r) = \begin{cases} [3/2 + 2c/3 - A](1 - r) - (1/2 + c/6)^2 + r[1/2 + c/3 - r/2] & \text{if } r \in (0, b_i^n] \\ [3/2 + 2c/3 - A - r/2](1 - r) & \text{if } r \in (b_i^n, 1) \end{cases}$$

Accounting for $\Delta V_E(r) = \Delta \Pi_E(r) + \Delta CS_E(r)$, we further obtain

$$\Delta V_E(r) = \begin{cases} [1/4 + 2c/3 - c^2/9](1 - r) - (1/2 + c/6)^2 + r[1/2 + c/3 - r/2] & \text{if } r \in (0, b_i^n] \\ [1/4 + 2c/3 - c^2/9 - r/2](1 - r) & \text{if } r \in (b_i^n, 1). \end{cases}$$

(36)

To determine the sign of $\Delta V_E(r)$, it is useful to define

$$G(r) \equiv [1/4 + 2c/3 - c^2/9](1 - r) - (1/2 + c/6)^2 + r[1/2 + c/3 - r/2],$$

with $G'(r) = 1/4 - c/3 + c^2/9 - r$. Noting that $G'(0) = 1/4 - c/3 + c^2/9 > 0$ and $G'(b_i^n) = -1/4 - c/2 + c^2/9 < 0$ holds for any $c \in [3/4, 6 - \sqrt{27}]$, it follows from the first line in (36) that $G(0) = c/2 - 5c^2/36 > 0$ and $G(b_i^n) = (7c/12 - c^2/9)(1/2 - c/6) > 0$ imply a positive value of $\Delta V_E(r)$ for any $r \in (0, b_i^n)$. In addition, we can define $\tilde{G}(r) \equiv 1/4 + 2c/3 - c^2/9 - r/2$, with $\tilde{G}'(r) = -1/2 < 0$. Hence, noting that $\tilde{G}(1) = 3/16 > 0$ holds.
for any $c \in [3/4, 6 - \sqrt{27}]$, we can conclude from the second line of (36) that $\Delta V_E(r) > 0$ for any $r \in (b^n_i, 1)$. Putting together, we find that $\Delta V_E(r) > 0$ for any $r \in (0, 1)$. This completes the proof of Lemma 2. QED.

**Derivation of equ. (23)**

Using $\pi^n_W = (3 + c)^2/18$ and $\pi^n_E = (3 - c)^2/18$, according to Proposition 1 together with $\Pi^m = A - 1/16$, according to our analysis in subsection 3.2, profit gains equal

$$\Delta \Pi = \Pi^m - \Pi^n = A - 17/16 - c^2/9 \quad (37)$$

Furthermore, consumer surplus changes are given by

$$\Delta CS = \int_0^{1/2} [p^n_W - p^n_E + b^2 - (b - 1/4)^2]db + \int_{1/2}^{b^n_i} [p^n_W - p^n_E + b^2 - (b - 3/4)^2]db + \int_{b^n_i}^{1} [p^n_E - p^n_E + (1 - b)^2 - (b - 3/4)^2]db$$

Substituting $p^n_W = c/3 + 1, p^n_E = 2c/3 + 1$ and $b^n_i = 1/2 + c/6$, according to Proposition 1, as well as $p^m_W = p^m_E = A - 1/16$ from our analysis in subsection 3.2, gives

$$\Delta CS = 18/16 + c/2 - c^2/36 - A. \quad (38)$$

Noting $\Delta V = \Delta \Pi + \Delta CS$, (23) follows from (37) and (38). QED.

**Proof of Lemma 3**

Consider $r \in (0, 1)$. Then, Profit gains in the Western region equal $\Delta \Pi_W(r) = [A - 17/16 - c^2/9]r$ and consumer surplus changes are given by

$$\Delta CS_W(r) = \begin{cases} B_4(r) & \text{if } r \in (0, 1/2] \\ B_4(1/2) + B_5(r) & \text{if } r \in (1/2, b^n_i] \\ B_4(1/2) + B_5(b^n_i) + B_6(r) & \text{if } r \in (b^n_i, 1) \end{cases}$$
with

\[ B_4(r) = \int_0^r (p_W^n - p_W^m + b^2 - (b - 1/4)^2) \, db, \quad B_5(r) = \int_{1/2}^r (p_W^n - p_E^m + b^2 - (b - 3/4)^2) \, db, \]

\[ B_6(r) = \int_{b_i^n}^r (p_E^n - p_E^m + (1-b)^2 - (b - 3/4)^2) \, db. \]

Substituting \( p_W^n = c/3 + 1, \ p_E^n = 2c/3 + 1 \) and \( b_i^n = 1/2 + c/6 \), according to Proposition 1, as well as \( p_W^m = p_E^m = A - 1/16 \) from our analysis in subsection 3.2, we further obtain

\[ \Delta CS_W(r) = \begin{cases} [1 + c/3 - A + r/4]r & \text{if } r \in (0, 1/2] \\ [1/2 + c/3 - A + (3/4)r]r + 1/8 & \text{if } r \in (1/2, b_i^n] \\ [3/2 + 2c/3 - A - r/4]r - 1/8 - c/6 - c^2/36 & \text{if } r \in (b_i^n, 1) \end{cases} \] \quad (39)

Noting \( \Delta V_W(r) + \Delta \Pi_W(r) + \Delta CS_W(r) \), we can therefore conclude that welfare changes in the Western region are given by

\[ \Delta V_W(r) = \begin{cases} [-1/16 + c/3 - c^2/9 + r/4]r & \text{if } r \in (0, 1/2] \\ [-9/16 + c/3 - c^2/9 + (3/4)r]r + 1/8 & \text{if } r \in (1/2, b_i^n] \\ [7/16 + 2c/3 - c^2/9 - r/4]r - 1/8 - c/6 - c^2/36 & \text{if } r \in (b_i^n, 1] \end{cases} \] \quad (40)

To determine the sign of \( \Delta V_W(r) \) let us first consider interval \((0, 1/2]\. Then, it follows from the first line in (40) that \( c \geq 6/4 - \sqrt{27}/4 \) ensures a positive value of \( \Delta V_W(r) \) for any \( r \in (0, 1/2]\. However, if \( c < 6/4 - \sqrt{27}/4 \) there exists a critical \( \tilde{r}_W^2 \equiv 1/4 - 4c/3 + 4c^2/9 \), such that \( \Delta V_W(r) > 0 \) if \( r \in (r_W^2, 1/2] \), while \( \Delta V_W(r) < 0 \) if \( r \in (0, r_W^2) \). Second, defining

\[ H(r) \equiv 1/8 + [-9/16 + c/3 - c^2/9 + (3/4)r]r, \]

with \( H'(r) > 0 \) for any \( r \in (1/2, b_i^n] \), we can conclude that \( H(1/2) = 1/32 + c/6 - c^2/18 > 0 \) implies \( \Delta V_W(r) > 0 \) for any \( r \in (1/2, b_i^n] \), according to the second line of (40). Third, we can define

\[ \bar{H}(r) \equiv [7/16 + 2c/3 - c^2/9 - r/4]r - 1/8 - c/6 - c^2/36, \]

with \( \bar{H}'(r) = 7/16 + 2c/3 - c^2/9 - r/2 \). Evaluating \( \bar{H}'(r) \) at \( r = b_i^n \) and \( r = 1 \), we further obtain \( \bar{H}'(b_i^n) = 3/16 + 7c/12 - c^2/9 > 0 \) and \( \bar{H}'(1) = -1/16 + 2c/3 - c^2/9 > 0 \).
that $\Delta H$ confirms that $\Delta H = c >,.<3(1-\sqrt{15/16})$.\textsuperscript{17} In any case, it follows from $\bar{H}(b^n_i) = 1/32 + 19c/96 + c^2/48 - c^3/54 > 0$ and $\bar{H}(1) = 1/16 + c/2 - 5c^2/36 > 0$ that $\Delta V_W(r) > 0$ if $r \in (b^n_i, 1)$. Putting together, we can conclude that $\Delta V_W(r) > 0$ for any $r \in (0, 1)$ if $c \geq 6/4 - \sqrt{27}/4$, while $c < 6/4 - \sqrt{27}/4$ implies that $\Delta V_W(r) >,.<0$ if $r >,.=<r_W^n.$

Let us now turn to the Eastern economy. From (23) and the first line in (40), it follows that $\Delta V_E(r) = \Delta V - \Delta V_W(r) > 0$ holds for any $r \in (0, 1/2] \text{ if } 1/16 + c/2 - 5c^2/36 > \Delta V_W(1/2)$. Noting $\Delta V_W(1/2) = 1/32 + c/6 - c^2/18$, therefore confirms $\Delta V_E(r) > 0$ for any $r \in (0, 1/2]$. Furthermore, as $H'(r) > 0$ for any $r \in (1/2, b^n_i]$, we can conclude that $\Delta V_E(r) = \Delta V - \Delta V_W(r) > 0$ holds for any $r \in (1/2, b^n_i]$, if $1/16 + c/2 - 5c^2/36 > \Delta V_W(b^n_i)$. Substituting $\Delta V_W(b^n_i) = \bar{H}(b^n_i) = 1/32 + 19c/96 + c^2/48 - c^3/54 > 0$, the latter condition is equivalent to $1/32 + 29c/96 - 23c^2/144 + c^3/54 > 0$. Noting $c \in (0, 6 - \sqrt{27})$, this confirms that $\Delta V_E(r) > 0$ holds for any $r \in (1/2, b^n_i]$. We can therefore focus on interval $(b^n_i, 1)$ in the subsequent analysis.

Profit changes in the Eastern economy equal $\Delta \Pi_E(r) = [A - 17/16 - c^2/9](1-r)$, while consumer surplus changes are given by

$$\Delta CS_E(r) = \int_r^1 [p^n_W - p^n_E + (1-b)^2 - (b - 3/4)^2] db$$

if $r \in (b^n_i, 1)$. Substituting $p^n_W = c/3 + 1$, $p^n_E = 2c/3 + 1$ and $b^n_i = 1/2 + c/6$, according to Proposition 1, as well as $p^n_W = p^n_E = A - 1/16$ from our analysis in subsection 3.2, implies

$$\Delta CS_E(r) = [5/4 + 2c/3 - A - r/4](1-r) \tag{41}$$

if $r \in (b^n_i, 1)$. Substituting into $\Delta V_E(r) = \Delta \Pi_E(r) + \Delta CS_E(r)$, we further obtain

$$\Delta V_E(r) = [3/16 + 2c/3 - c^2/9 - r/4](1-r) \tag{42}$$

if $r \in (b^n_i, 1)$. We can now define $I(r) \equiv 3/16 + 2c/3 - c^2/9 - r/4$, so that $I(r)(1-r) = \Delta V_E(r)$ if $r \in (b^n_i, 1)$. Noting that $I(1) = \bar{H}'(1)$, we can conclude that $\Delta V_E(r) > 0$ for any $r \in (b^n_i, 1) if c \geq 3(1 - \sqrt{15/16})$. In contrast, $c < 3(1 - \sqrt{15/16})$ implies that there exists

\textsuperscript{17}Consider $d\bar{H}'(1)/dc > 0$. Then, $c^2 - 6c + 9/16$ gives a critical $\tilde{c} = 3 - 3\sqrt{15/16} \in (0, 1)$, such that $\bar{H}'(1) >,.<0$ if $c >,.=,<\tilde{c}$. 

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a critical $r_E^2 \equiv 3/4 + 8c/3 - 4c^2/9$, such that $\Delta V_E(r) > 0$ if $r \in (b_i^n, r_E^2)$, while $\Delta V_E(r) < 0$ if $r \in (r_E^2, 1)$. Putting together, we can therefore derive the following conclusion: If $c \geq 3(1 - \sqrt{15/16})$, then $\Delta V_E(r) > 0$ for any $r \in (0, 1)$. However, if $c < 3(1 - \sqrt{15/16})$, then $\Delta V_E(r) > 0$ for any $r \in (0, r_E^2)$, while $\Delta V_E(r) < 0$ if $r \in (r_E^2, 1)$. This completes the proof of Lemma 3. QED.

References


