

Hysteresis in Export Markets *

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Abstract

This paper develops a dynamic monopolistic competition model with heterogeneous firms to analyze the effects of uncertainty on international trade. We characterize a stationary equilibrium, with multiple countries, where firms' productivities evolve stochastically over time. Our model retains the main results of previous recent papers like Melitz (2003) and Bernard, Eaton, Jensen and Kortum (2003) and provides additional new predictions. The model is mostly in closed-form and therefore very amenable to estimation and simulation, representing a useful tool for analyzing the effects of trade policies. We provide some initial evidence using plant-level data for Chile and Colombia to show the importance of the model. Several moments, like average age, size and productivity of different categories of firms (exporters, entrants, exiters, incumbents), the survivor function or transition matrices for productivity or export participation are derived and can be used to match static and dynamic features of the data. The introduction of reentry export costs generate hysteresis in export participation creating a band of coexistence within the stationary distribution of firms'

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productivities. We derive the density of exporters along the band of inaction in a way that is potentially interesting on its own and could be applied to other economic problems. The decision to export becomes history-dependent and new entrants and incumbent firms might sustain temporary negative profits before becoming profitable.

1 Introduction

Recent empirical studies, like Bernard and Jensen (1995), Clerides et al. (1998) and Anderson (2005) suggest that successful theoretical frameworks for studying firms and the decision to export should incorporate intraindustry heterogeneity in size and productivity and take into account international trade costs. Firms are heterogeneous in many respects: firms' productivities differ widely even within narrowly defined industries; exporting firms are, on average, more productive and bigger than nonexporting counterparts. International trade costs are large, even between apparently highly integrated economies. Some costs are increasing in the amount shipped while other costs are fixed and occur every time the firm tries to enter in a foreign market.

In the past couple of years, the literature on international trade has successfully addressed some of these issues: Bernard, Eaton, Jensen and Kortum (2003)¹ develop a Ricardian model of plant-level export behavior while Melitz (2003) provides a model based on monopolistic competition. These models represent an important step in reconciling macro- with micro-level trade data but they are static. Therefore, they don't address dynamic features of the data like the fact that firm productivity, size and exporting status change over time; firms' decisions to export do not only depend on current productivity; exporting firms coexist with nonexporting firms even if they are observationally equivalent in terms of productivity, size or other characteristics. Moreover, they do not rationalize the existence of a measure of exporters that are less productive than nonexporters or the fact that entrants or even incumbents might sustain temporary negative profits before becoming profitable.

On the other side, the literature on firm dynamics has developed closed economy models which allow for heterogeneity in the characteristics of firms. The landmark paper is Hopenhayn (1992) while a recent contribution in this respect is the work by Luttmer (2004) who proposes an analytically tractable model of balanced growth that allows for extensive heterogeneity in the technologies used by firms.

¹BEJK from here on.

Building upon Melitz (2003) and Luttmer (2004), we construct a dynamic general equilibrium model of trade among multiple countries that combines the following four key ingredients: 1) firms have heterogeneous productivity; 2) compete globally to sell their differentiated product and 3) firm-level productivity evolves over time stochastically; 4) entry (or reentry) into the export market requires the firm to sustain a sunk cost.

Entrepreneurs make an initial investment to set up these firms and draw their initial productivity level from a common distribution. Production for the domestic market starts even if (unlike in Melitz (2003)) initial profits are negative (as long as they are not too negative) and continues until the sum of current profits and the value of the option to exit are high enough. If firm productivity exceeds a cutoff level it becomes profitable to enter foreign markets. In order to do that, the firm must incur a sunk cost which can be interpreted as the cost of establishing distribution channels, learning about the foreign markets preferences and standards and adapting to them, updating old export products. If, later on, productivity falls under the level at which the firm started exporting, the entrepreneur prefers to keep exporting, as long as the value of current exporting profits plus the value of the option to stop exporting is bigger than the value of the option of reentering the export market.

In equilibrium, in every country, 1) consumers maximize their intertemporal utility by choosing a sequence of consumption of a composite good, made of available domestic and foreign varieties, subject to the intertemporal budget constraint 2) there is a closed-form stationary distribution of firms' productivities and within it a band of inaction; 3) while the distribution is stationary, new firms enter, incumbent firms become more or less productive, export or stop exporting, eventually exit; 4) labor and goods markets clear.

Our model is easily amenable to estimation. Closed-form solutions allow the derivation of several static and dynamic moments that are useful for matching the model to the data: average size, productivity and age of incumbents, entrants, exiters, exporters or nonexporters, the hazard function for exiting or for becoming an exporter

represent some examples.²

As in Melitz (2003) and BEJK (2003), firms that are more productive are bigger, both in terms of output and revenues and are more likely to export. Unlike the previous literature, our model generates hysteresis in export participation in a general equilibrium framework. Hysteresis is defined, in our context, means that export participation is history-dependent. The presence of hysteresis has been documented by, among others, Roberts and Tybout (1997) and is important in understanding why trade policies might have different effects in different countries or in the same country at different stages of its evolution and why even temporary policies might have permanent effects. The presence of hysteresis also implies that, as in the data³, some exporters are less productive than some nonexporters. Our model departs from Melitz (2003) also in predicting that new firms might sustain initial negative profits and, only later on, become profitable. Again consistently with the data, the stationary productivity distribution follows a Pareto density in the upper tail but is increasing in the lower tail, implying that there are fewer small firms than would be the case if Zipf's law was held.⁴ Extending Luttmer's framework, we derive the stationary density of exporters along the band of inaction as the solution of two linked partial differential equations. Our solution technique could prove useful to analyze other economic problems.

The structure of the paper is as follows. In the next section, we discuss some facts about firm productivity, profitability, entry and, export participation using Chilean data. In section 3, we describe the model under autarky. In section 4, we introduce trade. First, we solve for the equilibrium assuming that a firm can costlessly enter and exit the export market. Then, we show that entry (and reentry) costs generate

²Calibration and simulation of the model with Chilean and Colombian data are the subject of ongoing research.

³See Irrarazabal and Opromolla (2005) for Chile, BEJK (2003) for the USA.

⁴In terms of the distribution, this means that the probability that the size (that in our model is directly connected to productivity) of a firm is greater than some z^* is proportional to $1/z^*$: $P(z > z^*) = \alpha/(z^*)^\theta$, with $\theta \simeq 1$. See Simon and Bonini (1958), Gabaix (1999), Axtell (2001) and Cabral and Mata (2003) for discussions of firm size distribution, Zipf's law and its empirical evidence.

hysteresis. In section 5 we simulate the model and show preliminary results on the effects of a reduction in trade barriers in the context of the benchmark open economy case. Finally section 6, concludes and proposes some extensions to the present framework.

2 Facts

We use data for two Latin American countries, Chile and Colombia, that underwent major trade liberalizations. Data for Chile comes from the "Encuesta Nacional Industrial Anual" (ENIA) conducted annually by Instituto Nacional de Estadística (INE), the Chilean Government's statistical office. ENIA is an unbalanced panel dataset covering all Chilean manufacturing plants with ten or more workers. The dataset extends from 1979 to 1996 and includes information on approximately 11,000 plants altogether, with about 4,800 plants per year. Data for Colombia comes from the Colombian Industrial Survey (CIS) conducted annually by Departamento Administrativo Nacional de Estadística, the Colombian Government's statistical office. CIS is an unbalanced panel dataset that extends from 1977 to 1991, covering all the manufacturing plants from 1977 to 1982 and all the manufacturing plants with ten or more workers from 1983 to 1991, with an average of about 6,500 plants per year. Both datasets contain detailed information on production, value added, sales, employment and wages (both white-collar and blue-collar), exports, investment, depreciation, energy consumption, balance sheet information, and other plant characteristics. Data on plant-level exports were only collected after 1990 for Chile and after 1981 for Colombia.

We start by showing that plants relative productivity, measured as output per worker, changes considerably over time. Tables 1 and 2 show the average quintiles transition matrix for the period 1980-96 for Chile and for the period 1981-91 for Colombia. The tables show that in both countries more than 40% of the plants that belong to one of the first four quintiles in year t changes quintile or exit in year

Table 1: Productivity Transition Matrix, Quintiles, Chile, Average 1980-96

		year $t + 1$					<i>Out</i>
		1	2	3	4	5	
year t	1	.64	.16	.04	.02	.00	.15
	2	.19	.49	.19	.04	.01	.08
	3	.04	.23	.48	.18	.01	.06
	4	.02	.05	.20	.57	.10	.05
	5	.00	.01	.02	.12	.81	.04

Table 2: Productivity Transition Matrix, Quintiles, Colombia, Average 1981-91

		year $t + 1$					<i>Out</i>
		1	2	3	4	5	
year t	1	.63	.13	.02	.01	.01	.20
	2	.16	.51	.16	.03	.01	.13
	3	.02	.20	.52	.15	.02	.10
	4	.01	.03	.18	.60	.10	.08
	5	.00	.00	.01	.12	.80	.06

$t + 1$. The overall transition pattern is very similar in the two countries with the only difference being a higher exit rate in Colombia.

In spite of the large time-series variation in productivity, firms are very persistent in terms of export participation. Table 3 shows the average transition matrix between export and nonexport status for the period 1990-1996 in Chile.

Table 3 suggests the presence of a high degree of persistence in export status. Out of 100 plants that export in year t , 84 plants keep exporting the next year, 12 plants stop exporting but keep producing for the domestic market and 4 plants becomes inactive. Among plants not exporting at t , 4% start exporting at $t + 1$, 89% keep selling their product only on the domestic market and 7% shut down. We use this matrix to perform a simple test to check if the data are consistent with ergodicity

Table 3: Export Status Transition Matrix, Chile, Average 1990-96

	export in $t + 1$	do not export in $t + 1$	exit in $t + 1$
export in t	.84	.12	.04
do not export in t	.04	.89	.07

and find that the average ratio between the number of exporters and the number of nonexporters is about 3.8, is declining over time and is quite close to the ergodic ratio of 2.8.

Exporters are on average more productive than nonexporters. Figure (A), from Irarrazabal and Opromolla (2005), and Figure (B) show the histogram of productivity by export status for Chile and Colombia. Productivity is measured as value added per worker and is normalized using the same overall (exporters and nonexporters) mean productivity. The distribution of exporters productivity is shifted to the right with respect to the productivity distribution of non-exporting plants. Value added per worker at the average exporting Chilean plant is 85 percent higher than at the average plant that does not export. Exporters' productivity advantage over non-exporters was estimated to be 33% in the US (BEJK) and 12.5% in France (Eaton et al. 2005). Even though average productivity is higher for exporters there is a consistent measure of exporters that are less productive than nonexporters.

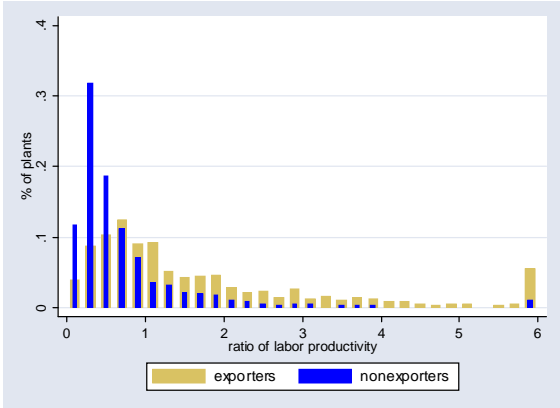


Figure A : Ratio of Plant Labor Productivity to Overall Mean: Exporters vs. Nonexporters, Chile 1992

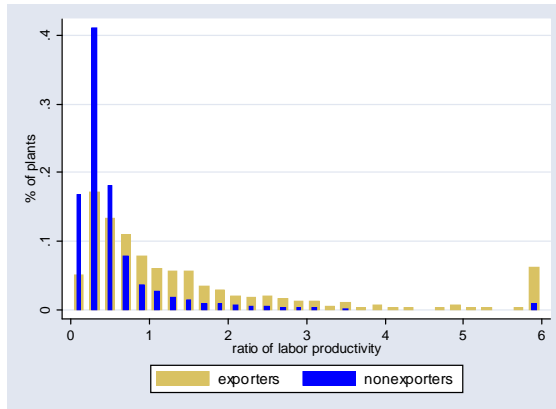


Figure B : Ratio of Plant Labor Productivity to Overall Mean: Exporters vs. Nonexporters, Colombia, 1988

Table 4: Productivity Difference When Plants Start and Stop Exporting

Year	Ln Productivity			p-value ⁽¹⁾ ($H_a: diff = 0$)	p-value ⁽¹⁾ ($H_a: diff > 0$)
	When Start Exporting	When Stop Exporting	Productivity Ratio ⁽²⁾		
1991	8.54	8.39	1.16	.28	.14
1992	8.70	8.61	1.09	.51	.26
1993	8.94	8.64	1.35	.04	.02
1994	8.97	8.71	1.30	.05	.02
1995	9.14	9.00	1.15	.34	.17
1996	9.32	9.23	1.09	.50	.25

Note: (1) Two-sample t test with equal variances; H_0 : the difference between $\ln(\text{productivity})$ when start exporting and $\ln(\text{productivity})$ when stop exporting is equal to zero; (2) $\exp(\ln \text{prod}_{start} - \ln \text{prod}_{stop})$;

Table 5: Size Difference When Plants Start and Stop Exporting

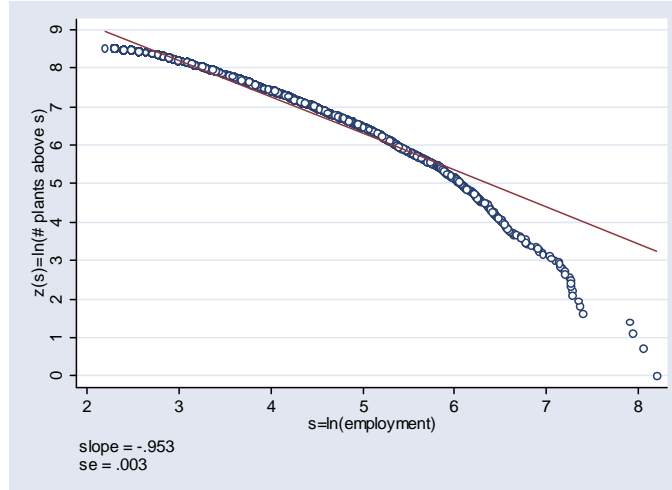
Year	Ln Sales			p-value ⁽¹⁾ ($H_a: diff = 0$)	p-value ⁽¹⁾ ($H_a: diff > 0$)
	When Start Exporting	When Stop Exporting	Sales Ratio ⁽²⁾		
1991	13.90	13.55	1.42	.09	.05
1992	13.77	13.71	1.06	.80	.40
1993	14.00	13.36	1.90	.01	.00
1994	14.06	13.32	2.10	.00	.00
1995	14.28	14.12	1.17	.49	.25
1996	14.23	14.01	1.25	.27	.14

Note: (1) Two-sample t test with equal variances; H_0 : the difference between $\ln(\text{sales})$ when start exporting and $\ln(\text{sales})$ when stop exporting is equal to zero; (2) $\exp(\ln \text{sales}_{start} - \ln \text{sales}_{stop})$;

Tables (4) and (5) compare the average productivity and size of plants that just started exporting (new exporters) and of plants that just stopped exporting (old exporters). We find some evidence in support of the hypothesis that plants decisions to enter and exit the export market do not rely on the same reference productivity level. New exporters' average productivity is always higher than average productivity of old exporters, even though the difference passes a two-sample t-test only during part of the sample period.

Figure (C) shows the size distribution of plants, measured in terms of employment, in 1992. The size distribution follows a Power Law with exponent equal to one (Zipf's' Law) if the plot of the natural logarithm of the number of plants above some size level s against the natural logarithm of plants labor force results in a straight line

with slope equal to minus one.⁵ The data follow Zipf's law quite closely for most of the size range with two exceptions: there are fewer very small and very big firms than what would be consistent with Zipf's law. Keeping in mind these and other features of the data, we now proceed to outline the model.



⁵The size s of a plant is distributed as a power law with exponent α and minimum size s_0 if the density of s is $f(s) = \alpha s_0^\alpha s^{-\alpha-1}$ ($s \geq s_0$, $\alpha > 0$). Suppose that N is the number of plants. The rank of all firms in the sample is $r(s) = N \left(\frac{s_0}{s}\right)^\alpha$ where the rank is decreasing in the size of the plant. Taking natural logs leads to $\ln r(s) = c - \alpha \ln(s)$ where $c = \ln N + \alpha \ln(s_0)$.

3 Set Up of the Model

3.1 Consumers

Time is continuous. Let Ω_h be the set, of measure L , of infinitely-lived consumers. The economy consists of one sector that produces differentiated products. Each consumer is endowed with one unit of labor at every point in time that is supplied inelastically. The wage is normalized to one. Goods are perishable and hence can only be used for consumption. Following Dixit and Stiglitz (1977), the representative consumer has preferences over sequences of a composite good $\{C_t\}_{t \geq 0}$ given by

$$E \left[\int_0^\infty e^{-\rho t} U(C_t) dt \right], \quad (1)$$

where ρ is the time-discounting rate and

$$C_t = \left(\int c_t(u)^{\frac{\sigma-1}{\sigma}} du \right)^{\frac{\sigma}{\sigma-1}}$$

where $\sigma > 1$ is the elasticity of substitution between any two differentiated goods.

The representative consumer chooses c_t to minimize the cost of acquiring C_t given a standard present-value budget constraint. Wealth consists of claims to firms and labor income. As a result,

$$c_t(u) = C_t \left[\frac{p_t(u)}{P_t} \right]^{-\sigma} \quad (2)$$

with the corresponding consumption-based price index

$$P_t = \left[\int p_t(u)^{1-\sigma} du \right]^{\frac{1}{1-\sigma}} \quad (3)$$

3.2 Firms

There is a continuum of firms, each choosing to produce a distinct variety of the goods using labor as input.⁶ At age a , a firm employs l_a units of labor to produce $e^{z_a} l_a$ units of the goods. We will refer to z_a as the productivity of the firm.

⁶Since we will consider a stationary equilibrium where all aggregate variables are constant, from now on we drop the time subscript in order to simplify the notation.

All firms share the same overhead per period fixed cost $f_d > 0$, constant over time. A firm must exit if this fixed cost is not paid. Exit is irreversible. Productivity evolves, independently across firms, according to a Brownian motion with drift α and diffusion coefficient ξ ,

$$dz_a = \alpha da + \xi dB_a \quad (4)$$

where $\{B_a\}_{a \geq 0}$ is a Wiener process. The initial productivity of a firm \bar{z} is drawn from a time-invariant probability density $g(\cdot)$. After entry, the trend of log productivity is determined by α .

A monopolistic competitive producer with productivity z prices its product according to the rule

$$p(z) = \frac{\bar{m}}{e^z} \quad (5)$$

where $\bar{m} = \sigma/(\sigma - 1)$ is the fixed markup and $1/e^z$ is the marginal cost of production. Firm revenue is

$$r(z) = R \left(\frac{p(z)}{P} \right)^{1-\sigma}, \quad (6)$$

where R is aggregate expenditure on the composite good. Firm profits are then

$$\pi(z) = r(z) - l(z) - f_d = \frac{r(z)}{\sigma} - f_d,$$

where $\frac{r(z)}{\sigma}$ represents variable profits. Using (5) and (6) we can see that profits also depend also on the aggregate price and on revenues:

$$\pi(z) = \frac{R}{\sigma} (\bar{m}^{-1} P e^z)^{\sigma-1} - f_d \quad (7)$$

A more productive firm charges a lower price (since marginal costs are lower and the markup is constant), is bigger both in terms of output and revenues (since lower price means higher demand and demand is elastic), and earns higher profits than a less productive firm (since variable profits are a constant fraction of firm's revenues).

Note that the productivity cutoff z_0 at which current profits are zero is

$$e^{z_0} = \frac{\bar{m}}{P} \left(\frac{\sigma f_d}{R} \right)^{\frac{1}{\sigma-1}}$$

This cutoff is decreasing in the price level P (since the wage and the markup are the same for every firm, the price index is really a measure of the degree of competition faced by the firm) and in the level of expenditure R . It is increasing in the fixed cost f_d and in the elasticity of substitution σ (provided that $R > \sigma f_d$, when goods become more substitutable, price deviations from the general price have bigger effects).

Using the price rule (5), the demand equation (2), and the production function, we can find the firm labor demand

$$l(z) = C(e^z)^{\sigma-1} \bar{m}^{-\sigma} P^\sigma, \quad (8)$$

which turns out to be a fraction $\frac{1}{\bar{m}}$ of the revenues

$$l(z) = \frac{1}{\bar{m}} r(z)$$

3.3 Entry, Exit and the Stationary Distribution

In this section we derive the stationary distribution of firms' productivities. We depart from previous models used in international trade (see Melitz (2003), Chaney (2005), and Eaton and Kortum (2002)) by using a model of industry equilibrium with dynamic stochastic productivities similar to the one of Luttmer (2004). Contrary to Melitz (2003), firms are subject to both ex-ante and ex-post uncertainty. Entry requires the entrepreneur to incur a sunk cost before knowing the initial productivity level. If the initial productivity is high enough, the firm starts producing. After that, productivity evolves over time according to (4) and the firm remains active until its value is positive. A positive drift coupled with a positive diffusion coefficient in (4) makes it worthwhile to keep producing even if current profits are negative. However, if profits become too negative or if the firm receives a bad shock uncorrelated to productivity, exit takes place. We begin by deriving the value of a firm as a function of its productivity level and we then proceed by deriving the ergodic productivity distribution.

3.3.1 Entry and Exit

Incumbent firms, indexed by $i \in \Omega_I$, exit the industry when their productivity falls below some threshold so that they are not able to cover the fixed cost of production f_d . Exit is irreversible. Firms exit when productivity is too low or because of bad shocks uncorrelated with productivity: these occur each period with an exogenous probability κ . This allows the model to capture exit uncorrelated to productivity and better match the data. A positive κ also allows for a stationary productivity distribution compatible with a positive drift in the stochastic process for productivity. We now derive the value of a firm.

The value function of a firm with productivity z can be expressed as the sum of operating profits over the interval $(t, t + dt)$ and the continuation value beyond $t + dt$. The value for a firm that discounts flows at the interest rate r and is subject to the exogenous exit rate κ is

$$V_d(z) = \left\{ \pi(z)dt + e^{-(r+\kappa)dt} E [V_d(z + dz|z)] \right\}.$$

Using Ito's lemma and equation (7) for firm's profits, we obtain an ordinary differential equation in the range of z where a firm is not shut down:

$$(r + \kappa)V_d(z) = \left(\frac{R}{\sigma} (\bar{m}^{-1} P e^z)^{\sigma-1} - f_d \right) + \alpha V_d'(z) + \frac{1}{2} \xi^2 V_d''(z).$$

Because of the fixed operating cost, it is optimal to shut down when productivity falls below some threshold z_d . The value of a firm must be zero at that point implying the value matching and smooth-pasting conditions

$$V_d(z_d) = 0$$

and

$$V_d'(z_d) = 0.$$

These conditions provide a complete characterization of the optimal policy of an active firm, the associated value function, and the critical value z_d .⁷ With these boundary

⁷See Stokey (2005) "Brownian Models in Economics" lectures notes for more details.

conditions, the value of a firm with productivity z is

$$V_d(z) = \frac{f_d}{(r + \kappa)} \frac{\beta_2}{\beta_2 - (\sigma - 1)} \left[e^{(\sigma-1)(z-z_d)} - \frac{\beta_2 - (\sigma - 1)}{\beta_2} - \frac{\sigma - 1}{\beta_2} e^{\beta_2(z-z_d)} \right] \quad (9)$$

for $z > z_d$, where β_2 is the negative root of the characteristic polynomial (see the Appendix). The value of the firm is increasing in z and can be interpreted as the sum of two components: the first two terms inside the square brackets in $V_d(z)$ reflect the expected discounted present value of the profits flow while the third term represents the value of the option to exit when the productivity falls below z_d (see the Appendix). The present value of revenues, and therefore of variable profits, is finite if the effective discount factor, given by the sum of the interest rate and the exogenous probability κ , is greater than the drift of variable profits. The following assumption guarantees that this is the case:

Assumption 1: *The preferences and productivity parameter satisfy*

$$\rho + \kappa > \alpha(\sigma - 1) + \xi^2(\sigma - 1)^2/2.$$

The exit productivity threshold z_d is

$$e^{z_d} = \frac{\bar{m}}{P} \left(\frac{\sigma f_d}{R} \right)^{\frac{1}{(\sigma-1)}} \gamma_d, \quad (10)$$

where $\gamma_d = \left[\frac{-\beta_2 \frac{1}{2} \xi^2 (\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)}{\beta_2 - (\sigma-1)} \right]^{\frac{1}{(\sigma-1)}}$ is less than one when the productivity drift α is positive (see the Appendix). This implies that $z_d < z_0$. Incumbents and new entrants remain active even if profits are currently negative, as long as they are not too negative. This happens because for $z \in (z_d, z_0)$ the value of the firm is still positive.

Inspection of (10) reveals that the barrier is inversely related to the aggregate price level. This will be critical when we open the economy to international trade. Opening to trade opening will decrease the aggregate price level, therefore pushing the lower threshold z_d up. This will be interpreted as a selection effect associated with international competition.

Prior to entry firms are identical. To enter, they must make a sunk initial investment $f_e > 0$ (measured in units of labor). Firms then draw their initial productivity

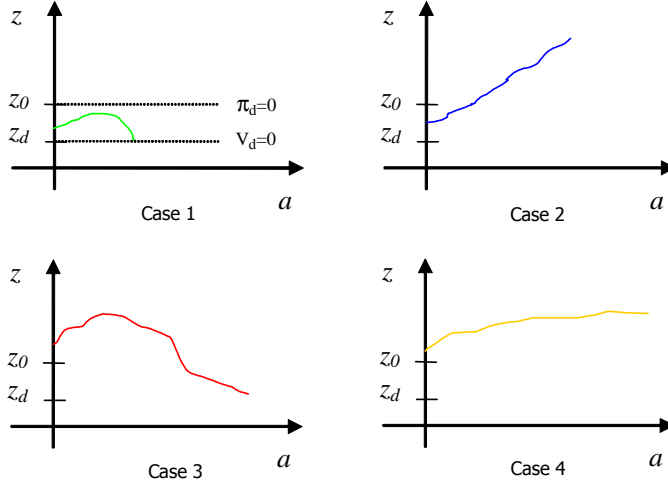


Figure 1: Examples of Feasible Productivity Paths

from a common distribution $g(\bar{z})$ with continuous cdf $G(\bar{z})$. Upon entry a firm may decide to exit immediately (and not pay the fixed cost f_d). New firms will keep trying to enter the industry until the expected value of a firm net of entry cost is zero, i.e. until the following free-entry condition is satisfied:

$$f_e = \int_{z_d}^{\infty} V_d(\bar{z}) dG(\bar{z}). \quad (11)$$

Figure (1) shows some sample productivity-age paths. The model allows for firms that start producing even if profits are negative, grow for a while and then become less efficient and exit the market (case 1) or keep growing, break even, and start making positive profits (case 2); obviously firms can also be profitable from the very beginning, grow for a while and take a decreasing path, eventually leading to exit (case 3) or grow for a while and then reach a stationary-looking productivity level (case 4).

3.3.2 The Stationary Distribution $p_u(z)$

As in Melitz (2003), in order to determine aggregate variables we need to characterize the invariant distribution of productivities. In the stationary equilibrium there is a

measure of firms M defined over the set of possible ages, initial and current productivities. The density of this measure is $Mp_u(a, z, \bar{z})$ where $p_u(a, z, \bar{z})$ is a probability density such that,

$$p_u(a, z, \bar{z}) = e^{-\kappa a} u(a, z | \bar{z}) g(\bar{z}) \frac{M_a}{M} \quad (12)$$

The main component of $p_u(\cdot)$ is $u(a, z | \bar{z})$, the transition density of a Brownian Motion (BM) like the one of equation (4), subject to a lower absorbing barrier $z_d < \bar{z}$. This is the solution of the following Kolmogorov equation,

$$\begin{cases} \frac{\partial u(a, z | \bar{z})}{\partial a} = -\alpha \frac{\partial u(a, z | \bar{z})}{\partial z} + \frac{1}{2} \zeta^2 \frac{\partial^2 u(a, z | \bar{z})}{\partial z^2} & z > z_d \\ u(a, z_d | \bar{z}) = 0 \text{ for } a \geq 0 ; u(0, z | \bar{z}) = \delta(z - \bar{z}), \end{cases}$$

where we impose the condition that the density be zero at the lower barrier z_d and that the initial productivity level is equal to \bar{z} .⁸ In order to derive $p_u(\cdot)$, we multiply the transition density by the initial productivity distribution $g(\cdot)$ in order to take into account firms with different initial productivity. The remaining two terms are related to exit and entry. The term $e^{-\kappa a}$ is the exogenous exit rate. Recall that the probability of exit due to hitting the lower barrier z_d is already factored in $u(a, z | \bar{z})$. Finally, the term $\frac{M_a}{M}$ is the entry rate that is determined by imposing that $p_u(\cdot)$ is a probability density over a, z and \bar{z} . The Appendix shows in detail how to derive the stationary density $p_u(\cdot)$.

Given the stationary probability density $p_u(a, z, \bar{z})$, we can determine the marginal probability distributions $p_u(a)$ (the age distribution of firms), $p_u(z)$ (the productivity distribution) and $p_u(\bar{z})$ (the distribution of initial productivity associated to the firms that are active in the steady state). In order to get a feeling for $p_u(z)$, it is useful to look at the simple case where all firms start with the same productivity level \bar{z} , i.e. to study the properties of $p_u(z | \bar{z})$. In the Appendix, we show that

$$p_u(z | \bar{z}) = \frac{\theta}{\theta + \theta_*} \left[\frac{\min \{ e^{(\theta_* + \theta)(z - z_d)}, e^{(\theta + \theta_*)(\bar{z} - z_d)} \} - 1}{e^{\theta(z - z_d)} \left(\frac{e^{\theta_*(\bar{z} - z_d)} - 1}{\theta_*} \right)} \right], \quad (13)$$

where θ and θ_* are both non-negative constants. The conditional density $p_u(z | \bar{z})$ is proportional to $e^{-\theta z}$ when $z > \bar{z}$ and therefore behaves, in the upper tail, as a

⁸ $\delta(\cdot)$ is the Dirac's delta function.

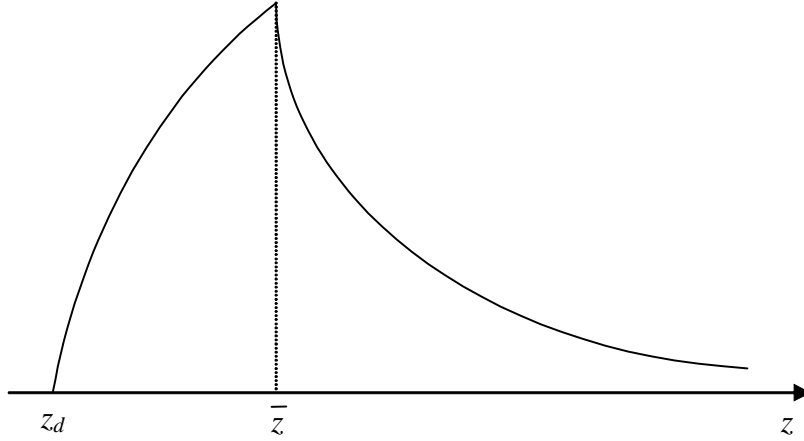


Figure 2: Equilibrium Probability Density of Productivity conditional on Initial Productivity \bar{z}

Power law with exponent θ for e^z , our productivity coefficient. When $z_d < z < \bar{z}$, the probability density is increasing in z (see Figure (2)). For the model to allow a stationary distribution with finite mean, we need to impose the following assumption:

Assumption 2: *The productivity parameters satisfy*

$$\kappa > \alpha + \xi^2/2.$$

This condition guarantees that $\theta > 1$ and that the mean of e^{z-z_d} is finite and can be expressed as

$$e^{\bar{z}-z_d} = \int_{z_d}^{\infty} e^{z-z_d} p_u(z|\bar{z}) dz = \frac{\theta}{(\theta-1)} \left[\frac{\frac{1-e^{-(1+\theta_*)(\bar{z}-z_d)}}{(\theta_*+1)}}{\frac{1-e^{-\theta_*(\bar{z}-z_d)}}{\theta_*}} \right] e^{\bar{z}-z_d}, \quad (14)$$

where the right hand-side is greater than $e^{\bar{z}-z_d}$ if $\alpha + \xi^2/2 > 0$. This means that the average firm is more productive than new entrants.

Note that the solution of the stationary problem is a probability density only for a particular value of the rate of entry. The rate of entry consistent with the stationary

equilibrium is

$$\frac{M_a}{M} = \left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\kappa} \right] g(\bar{z}) d\bar{z} \right]^{-1} \quad (15)$$

Note that we can now interpret the stationary solution under a different light,

$$p_u(a, z, \bar{z}) = p_u(a, z|\bar{z})p_u(\bar{z})$$

where

$$p_u(\bar{z}) = \frac{\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\kappa} g(\bar{z})}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\kappa} \right] g(\bar{z}) d\bar{z} \right]}$$

is the marginal probability density for the initial productivity \bar{z} in the stationary equilibrium. As Luttmer (2004) points out, this probability density is different from $g(\bar{z})$, the density of initial productivity among potential entrants, because of the ex-ante and ex-post selection processes. Ex ante selection requires firms to enter the market only if their initial productivity is bigger than z_d . Ex-post selection implies that firms with initial productivity close to the cutoff z_d have a higher probability of exiting due to a negative firm-specific shock and therefore they are less represented in $p_u(\bar{z})$: the term $1 - e^{-\theta_*(\bar{z}-z_d)}$ is increasing in \bar{z} . The effect of ex-post selection is stronger when exit uncorrelated to productivity is relatively less important (lower κ).

Finally, we are ready to determine an expression for the stationary distribution of productivities $p_u(z)$. In the appendix we show that

$$p_u(z) = \int_{z_d}^{\infty} p_u(z|\bar{z})p_u(\bar{z})d\bar{z} = \frac{\int_{z_d}^{\infty} p_u(z|\bar{z}) \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\kappa} \right] g(\bar{z})d\bar{z}}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\kappa} \right] g(\bar{z})d\bar{z} \right]}. \quad (16)$$

Note that $p_u(z)$ depends on the conditional probability density $p_u(z|\bar{z})$ and on the marginal $p_u(\bar{z})$. If $g(\cdot)$ is a degenerate distribution with all the mass at some point \bar{z} , then $p_u(z|\bar{z})$ itself is the equilibrium firm productivity density. The stationary productivity distribution is consistent with the data showed in section 2 if $g(\cdot)$ is a

distribution with few firms that are much larger than z_d . This expression is similar to equation (8) in Melitz (2003). In our setting though, as pointed out earlier, the equilibrium productivity distribution $p_u(z)$ differs from the productivity distribution of potential entrants because of both ex-ante and ex-post selection.

3.3.3 Taking the Model to the Data

Given the stationary distribution $p_u(a, z, \bar{z})$ we can compute several statistics about the characteristics of the firms. As in previous studies we can easily derive the average productivity of firms \tilde{z} and, as a consequence, the average size \tilde{r} (in terms of revenue) of active firms,

$$\tilde{z} = \int_{z_d}^{\infty} z p_u(z) dz \quad ; \quad \tilde{r} = \int_{z_d}^{\infty} R \left(\frac{\bar{m}}{e^z P} \right)^{1-\sigma} p_u(z) dz.$$

Differently from those studies, we can look at the time dimension, and derive the average age of incumbents \tilde{a} ,

$$\tilde{a} = \int_0^{\infty} a p_u(a) da.$$

We can also compute the average revenue per age-category \tilde{r}_{a_1, a_2}

$$\tilde{r}_{a_1, a_2} = \int_{a_1}^{a_2} \int_{z_d}^{\infty} r(z) \frac{p_u(z|a) p_u(a)}{P_u(a_2) - P_u(a_1)} dz da \quad , \quad [a_1, a_2] \subseteq [0, \infty),$$

where $P_u(\cdot)$ is the cdf of $p_u(\cdot)$ and $p_u(z|a)$ is the productivity density conditional on age.

Knowledge of the stationary productivity distribution $p_u(z)$ and of the transition density $u(a, z|\bar{z})$ makes it very easy to track movements of firms in the steady state. In particular, we can sort firms in quantiles and compute the probability that a firm in quantile q_{ab} at period t_1 will move to quantile q_{cd} at some other period t_2 . The probability that a plant with productivity $z_1 \in [a, b]$ at period t_1 will have a productivity level $z_2 \in [c, d]$ at period t_2 is

$$\Pr [z_2 \in [c, d] | z_1 \in [a, b]] = \frac{\int_c^d \int_a^b p_u(x) u(t_2 - t_1, y|x) e^{-\kappa(t_2 - t_1)} dx dy}{\int_a^b p_u(x) dx}$$

Moreover, given the stationary productivity distribution and the relevant cutoff values we can derive the probability that a plant won't exit before t years, as a function of current age, that is the following survival function,

$$P(T(z_d) > t|a) = \int_{z_d}^{\infty} \int_{z_d}^{\infty} u(t, x|\bar{x}) e^{-\kappa t} p_u(\bar{x}|a) dx d\bar{x}.$$

where $p_u(\bar{x}|a)$ is the productivity stationary distribution conditional on age.

4 Equilibrium in a Closed Economy

A closed-economy stationary equilibrium is defined by sequences

$$\left[c^h(u) \right]_{u \in \Omega, h \in \Omega_h}, \left\{ \left[p^i(z_a) \right]_{a \geq 0}, \left[l^i(z_a) \right]_{a \geq 0}, d_{a \geq 0}^i \right\}_{i \in \Omega_I}$$

where (i) each consumer h chooses, in each period, from the set of available varieties Ω , to consume $c(u)$ units of good u to maximize the intertemporal utility function of equation (1) subject to an intertemporal budget constraint; (ii) incumbent firms choose, at every age a , price $p(z_a)$ and variable labor $l(z_a)$ in order to maximize profits, given by equation (7), taking the price index of the economy as given; (iii) incumbent firms, at every age a , decide if to keep being active ($d_a = 1$) or to exit ($d_a = 0$) knowing that productivity evolves according to the Brownian motion of equation (4); (iv) firms enter, if the productivity draw is bigger than z_d , until the free-entry condition (11) is satisfied; (v) labor and goods markets clear.

With an expression for the stationary distribution of productivity, we can proceed to solve for the stationary equilibrium under autarky. First, notice that solution of the consumer's intertemporal problem pins down the interest rate as

$$r = \rho.$$

We can determine the root of the characteristic polynomial β_2 and hence the only unknown in the value function $V_d(z)$ is the lower barrier z_d . The free entry condition (11) allows to solve for it. Then we can compute the equilibrium probability density

for the log of productivity $p_u(z)$. The price index becomes then an equation in only two unknowns, P and M :

$$P^{1-\sigma} = \int_0^M p(u)^{1-\sigma} du = M \int_{z_d}^{\infty} \left(\frac{\bar{m}}{e^z}\right)^{1-\sigma} p_u(z) dz.$$

The second equation in P and M comes from the labor market. Recall that labor is used to create new firms (L_e), in the form of fixed production costs (L_f) and in the form of variable production costs (L_p):

$$\begin{aligned} L_e &= M_a f_e \\ L_f &= M f_d \\ L_p &= MRP^{\sigma-1} \bar{m}^{-\sigma} \int_{z_d}^{\infty} (e^z)^{\sigma-1} p_u(z) dz. \end{aligned}$$

The expression for L_p is found by recalling the equation for variable labor demand (8) and by recalling that $R = CP$. Note that the number of entry attempts M_a is directly proportional to the equilibrium number of firms through (15) so that L_e is also proportional to M . Labor market clearing requires

$$L_p + L_f + L_e = L$$

and this is the second equation in M and P .⁹ Finally, we can determine R from z_d and P .

5 The Open Economy

Consider an economy consisting of N countries all with the structure described in section 3. We suppose that a firm in country i that exports to country j bears a fixed operating export cost f_x per foreign market. Goods that are exported are then subject, like in Melitz (2003) and BEJK (2003), to a melting transportation cost $\tau > 1$. That is, we assume that country i needs to ship τ units of the goods for one unit to arrive in country j .

⁹Note that knowledge of z_d implies knowledge of $RP^{\sigma-1}$.

5.1 The Case of No Entry Cost

We consider the economy in its stationary equilibrium and we drop the index for age and time. Like in the closed-economy equilibrium, a firm with productivity z operating in the domestic market will obtain profits

$$\pi_d(z) = \frac{R}{\sigma} (\bar{m}^{-1} P e^z)^{\sigma-1} - f_d$$

If the firm decides to export to any other single country, the firm will earn some *additional* profits

$$\pi_{ex}(z) = \tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1} P e^z)^{\sigma-1} - f_x \quad (17)$$

Assume that each firm can enter and exit from the export sector costlessly. This implies that period profits from exporting to another country are

$$\pi_{ex}(z) = \max \left[\tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1} P e^z)^{\sigma-1} - f_x, 0 \right]$$

We are going to derive the solution of the following non-homogeneous ordinary differential equation

$$-(r + \kappa)V_{ex}(z) + \frac{1}{2}\xi^2 V_{ex}''(z) + \alpha V_{ex}'(z) + \pi_{ex}(z) = 0.$$

Since the forcing function is defined differently when current profits are positive or negative we need to solve the equation separately for the two cases and then stitch together the two solutions at the point (z_{ex}) where profits from exporting are zero:

$$\pi_{ex}(z) = 0 \iff e^{z_{ex}} = \frac{\tau \bar{m}}{P} \left(\frac{\sigma f_x}{R} \right)^{\frac{1}{\sigma-1}}. \quad (18)$$

Since, in the data we observe that some firms don't export¹⁰, we will assume that $\left(\frac{\tau}{\gamma_d}\right)^{\sigma-1} > \frac{f_d}{f_x}$, which guarantees that $z_{ex} > z_d$. Note that this also implies that the least productive exporter is more productive than the most productive nonexporter (see Figure (3)). This is something that we do not usually observe in the data and that we will take into account in the cost-of-entry case.

¹⁰See Irarrazabal and Opromolla (2005) for Chile and BEJK (2003) for the USA.

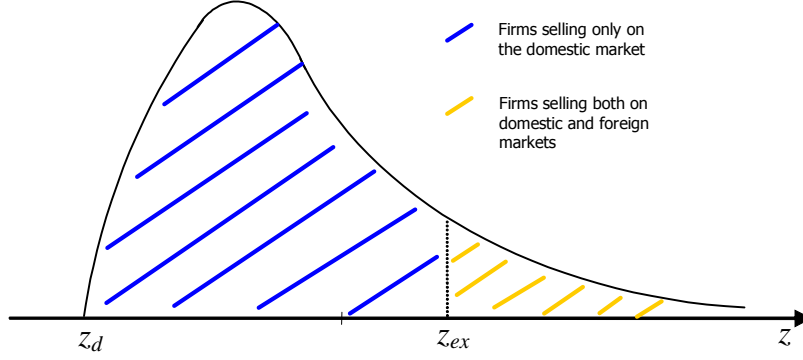


Figure 3: Firms' Productivity Distribution, No Cost of Entry in Foreign Markets

The value of the firm is $V_{>ex}(z)$ for the region where $z > z_{ex}$,

$$V_{>ex}(z) = \frac{f_x \left[e^{\beta_2(z-z_{ex})} \left(\frac{\sigma-1-\beta_1}{\beta_2-\beta_1\beta_2} \right) - e^{(\sigma-1)(z-z_{ex})} \right]}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} \quad (19)$$

$$- \frac{f_x}{(r+\kappa)} \left[\frac{\beta_1}{\beta_2 - \beta_1\beta_2} e^{\beta_2(z-z_{ex})} + 1 \right]$$

and $V_{<ex}(z)$ for the region where $z < z_{ex}$,

$$V_{<ex}(z) = \left\{ \begin{array}{l} \frac{f_x}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} \\ \frac{\beta_2(\sigma-2)}{\beta_1} - \beta_2 \frac{f_x}{(r+\kappa)} \end{array} \right\} \frac{e^{\beta_1(z-z_{ex})}}{\beta_2 - \beta_1\beta_2} \quad (20)$$

Note that the second and fourth terms in (19) represents the expected discounted profit flows from exporting, while the remaining two terms represent the value of the option to stop exporting. In the region (z_d, z_{ex}) , $V_{<ex}(z)$ represents instead the value of the option to resume exporting.

Having determined the cutoff z_{ex} , we can now draw on the stationary distribution analysis to derive expressions for the average age (\bar{a}_{ex}), productivity (\bar{z}_{ex}), and size (\bar{r}_{ex} , in terms of revenues) of exporters:

$$\tilde{a}_{ex} = \int_0^\infty a \int_{z_{ex}}^\infty \frac{p_u(a, z)}{\int_0^\infty \int_{z_{ex}}^\infty p_u(a, z) dz da} dz da; \quad \tilde{z}_{ex} = \int_{z_{ex}}^\infty z \frac{p_u(z)}{1 - P_u(z_{ex})} dz;$$

$$\tilde{r}_{ex} = \int_{z_{ex}}^\infty R \left(\frac{\bar{m}}{e^z P} \right)^{1-\sigma} \frac{p_u(z)}{1 - P_u(z_{ex})} dz.$$

These are the model equivalent of some of the statistics that we showed in Section 2.

Trade Equilibrium

To determine the equilibrium we assume that all fixed costs and the distribution $g(\bar{z})$ are identical for all countries. Once countries are allowed to trade, new firms will keep trying to enter the industry until the expected value of setting up a firm net of entry cost is zero, that is

$$f_e = \int_{z_d}^{z_{ex}} [V_d(\bar{z}) + (N-1)V_{<ex}(\bar{z})] dG(\bar{z}) + (N-1) \int_{z_{ex}}^{+\infty} [V_{>ex}(\bar{z}) - V_{<ex}(\bar{z})] dG(\bar{z}). \quad (21)$$

Note that the expected value of a firm include both the possibility that the firm will produce only for the domestic market and that the firm will also export to all the other $N-1$ countries. An open-economy, no-entry-cost, stationary equilibrium is defined by

$$\left[c^h(u) \right]_{u \in \Omega, h \in \Omega_h}, \left\{ [p^i(z_a)]_{a \geq 0}, [l^i(z_a)]_{a \geq 0}, [d_a^i]_{a \geq 0}, [e_a^i]_{a \geq 0} \right\}_{i \in \Omega_I}$$

where (i) each consumer h chooses, in each period, from the set of available varieties Ω , to consume $c(u)$ units of good u to maximize the intertemporal utility function of equation (1) subject to an intertemporal budget constraint; (ii) incumbent firms choose, at every age a , price $p(z_a)$ and variable labor $l(z_a)$ in order to maximize domestic profits, given by equation (7), and eventual exporting profits, given by equation (17), taking the price index of the economy as given; (iii) incumbent firms, at every age a , decides if to keep being active ($d_a = 1$) or to exit ($d_a = 0$) and if to export ($e_a = 1$) or not ($e_a = 0$) knowing that productivity evolves according to the Brownian motion of equation (4); (iv) firms enter, if the productivity draw is bigger than z_d , until the free-entry condition (21) is satisfied; (v) labor and goods markets clear.

Replacing the expressions for $V_d(z)$, $V_{<ex}(z)$, $V_{>ex}(z)$ in (21) and noting that $z_{ex} = \ln e^{z_d \frac{\tau}{\gamma_d}} \left(\frac{f_x}{f_d} \right)^{\frac{1}{\sigma-1}}$, yields an equation in the only unknown z_d . From here we can proceed exactly as in the close economy case.¹¹ Only the price index equation is

¹¹Note that now the demand for labor depends also on the fraction of

slightly different. All brands that are produced in a country have a consumer price $1/\bar{m}e^z$ and all imported brands have a consumer price of $\tau/\bar{m}e^z$ when the exporter productivity is z . This implies that the equation for the price index is

$$P^{1-\sigma} = \int_0^{M^*} p(u)^{1-\sigma} du = M \int_{z_d}^{\infty} \left(\frac{\bar{m}}{e^z}\right)^{1-\sigma} p_u(z) dz + (N-1)M \int_{z_{ex}}^{\infty} \left(\frac{\tau\bar{m}}{e^z}\right)^{1-\sigma} p_u(z) dz,$$

where M^* represents the number of firms supplying the market or the number of varieties offered.

Allowing for trade reduces the price level P and therefore induces an increase in the cutoff productivity level z_d . This result resembles those of other trade models with heterogenous firms. Firms who want to export need to pay a fixed cost f_x . This selects high productivity firms into the export market. Domestic firms with high enough productivity to survive pay higher wages. Since labor supply is inelastic, competition in the local labor market pushes up the real wage and forces low productivity firms to leave the industry.

5.2 The Case of Entry Cost: Hysteresis

We now analyze the conditions that determine the equilibrium with re-entry costs. It is well-known that when firms' productivities are stochastic, the introduction of an entry cost gives rise to hysteresis (see Dixit (1989), Baldwin (1989), Clerides et-al (1998)). To our knowledge, however, we are the first to model hysteresis in export markets in a general equilibrium framework.

Assume that each time a firm decides to enter or reenter the export market, it has to pay a sunk cost f_h . This cost can be interpreted as the cost of establishing distribution channels, learning about the foreign markets preferences and standards and adapting to them and updating old export products (see Roberts and Tybout (1997), Anderson and van Wincoop (2004)).

firms that are exporting, that is, $L_f = M \left[f_d + f_x \int_{z_{ex}}^{\infty} p_u(z) dz \right]$ and $L_p = MRP^{\sigma-1} \bar{m}^{-\sigma} \left[\int_{z_d}^{\infty} (e^z)^{\sigma-1} p_u(z) dz + (N-1) \tau^{-\sigma} \int_{z_{ex}}^{\infty} (e^z)^{\sigma-1} p_u(z) dz \right]$.

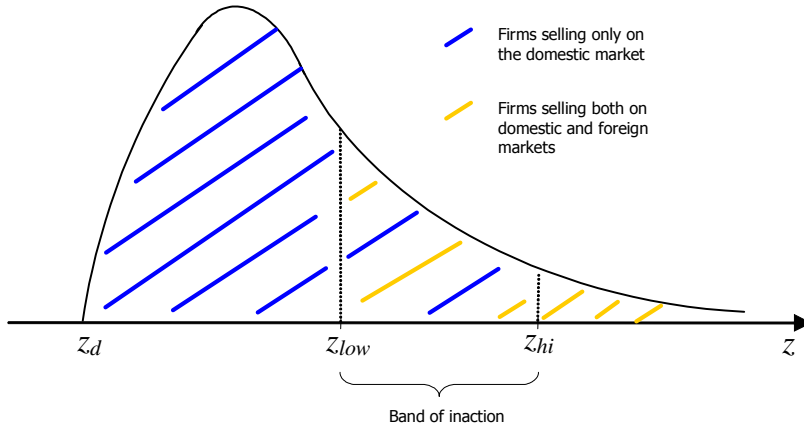


Figure 4: Firms' Productivity Distribution with Cost of Entry in Foreign Markets

The next proposition summarizes the optimal policy of a firm subject to entry and re-entry costs.

Proposition 1 *An optimal strategy is characterized by three thresholds $\{z_d, z_l, z_h\}$ with $z_d < z_l < z_h$ such that (i) a firm is active if $z > z_d$, (ii) a nonexporting firm will stay as a nonexporter as long as $z < z_h$, and (iii) an exporting firm will keep exporting as long as $z > z_l$. Furthermore, there is a band of inaction $z_l < z < z_h$ where an exporting plant will keep exporting and a nonexporting plant will decide not to enter the export sector.*

The previous proposition allows us also to state the following implication of the model which we observe in the data.

Corollary 2 *There is a positive measure of nonexporting firms that are more productive than some exporting firms*

These two propositions are illustrated in Figure (4).

As in the case without entry-cost we have that the value of a firm that only sells in the domestic market is given by

$$V_d(z) = \left\{ \pi_d(z) \Delta t + e^{-(r+\kappa)\Delta t} E[V_d(z + \Delta z|z)] \right\}.$$

On the other hand, if the firms is also exporting we must adjust our analysis with respect to the previous case. The value of the firm is now a function of two state variables, the productivity level z and a discrete state variable which indicates whether the firm is currently exporting or not. For a non exporting firm, in the region (z_d, z_h) we have

$$(r + \kappa)V_l(z) = \alpha V_l'(z) + \frac{1}{2}\xi^2 V_l''(z)$$

which has the general solution

$$V_l(z) = a_1 e^{\beta_1 z} + a_2 e^{\beta_2 z}$$

where a_1 and a_2 are constant to be determined and β_1 and β_2 are the roots of the quadratic equation determined earlier.

Since the option to export gets very far out of the money as z becomes lower and lower, the coefficient a_2 corresponding to the negative root β_2 should be set to zero. This leaves,

$$V_l(z) = a_1 e^{\beta_1 z}.$$

Next let's consider the value of an exporting firm in the region (z_l, ∞) . The ordinary differential equation is

$$(r + \kappa)V_h(z) = \pi_{ex}(z) + \alpha V_h'(z) + \frac{1}{2}\xi^2 V_h''(z)$$

where $\pi_{ex}(z)$ is given by equation (17).

The general solution to this equation (after setting the coefficient corresponding to the positive root to zero) is

$$V_h(z) = b_2 e^{\beta_2 z} - \frac{\tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1} P)^{\sigma-1}}{\frac{1}{2}\xi^2 (\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z} - \frac{f_x}{(r+\kappa)}.$$

The boundary conditions are

$$\begin{aligned} V_h(z_h) &= V_l(z_h) + f_h, \\ V_h'(z_h) &= V_l'(z_h), \\ V_h(z_l) &= V_l(z_l), \\ V_h'(z_l) &= V_l'(z_l). \end{aligned} \tag{22}$$

After replacing the expressions for $V_l(\cdot)$ and $V_h(\cdot)$, (22) gives a system of four equations in the four unknowns z_l , z_h , a_1 , and b_2 . The first and third conditions guarantee continuity in the function expressing the value of the firm. Note that, unlike in the previous case without cost of entry, the value of an exporting firm and the value of a nonexporting firm must be determined simultaneously. Due to high non-linearity in the system, it's difficult to obtain an analytical solution for the thresholds. A partial characterization is possible. First, the thresholds satisfy $0 < z_l < z_h < \infty$ and the coefficients a_1 and b_2 are positive. Second, suppose that the firm is not an exporter and that it believes that z will persist unchanged forever. The firm will decide to become an exporter if $\pi_{ex}(z) \geq (r + \kappa)f_h$. This is the exporting cutoff when there is no uncertainty and z is constant over time. In our case instead, $\pi_{ex}(z_h) > (r + \kappa)f_h > 0$ which means that z_h , the exporting cutoff, is larger than the productivity level at which the firm decides to become an exporter when there is no uncertainty and z is constant over time. This also implies that z_h is bigger than z_{ex} , the productivity cutoff for exporting in the no-entry-cost case. When domestic producers take into account the uncertainty over future profits, they are more reluctant to become exporters. Similarly, exporters are also more reluctant to abandon foreign markets. This is consistent with the difference between the average productivity for new and old exporters that we showed in Section 2. Third, the width of the band of inaction (z_l, z_h) is an increasing function of the sunk cost f_h . As the sunk cost f_h increases the lower cutoff, z_l , is decreasing while the upper cutoff, z_h , is increasing.

Figure (5) shows some sample paths for firms' productivity and export status: in case 1, a firm starts as a nonexporter but as productivity increases the firm becomes, at age a_h , an exporter. Case 2 portraits the evolution of a firm that exports from the very beginning but is on a decreasing productivity path and eventually stops exporting at age a_l . Finally, in case 3, an initially nonexporting firm starts exporting when productivity overtakes z_h at age a_h but then receives some bad shocks and stop exporting at age a_l .

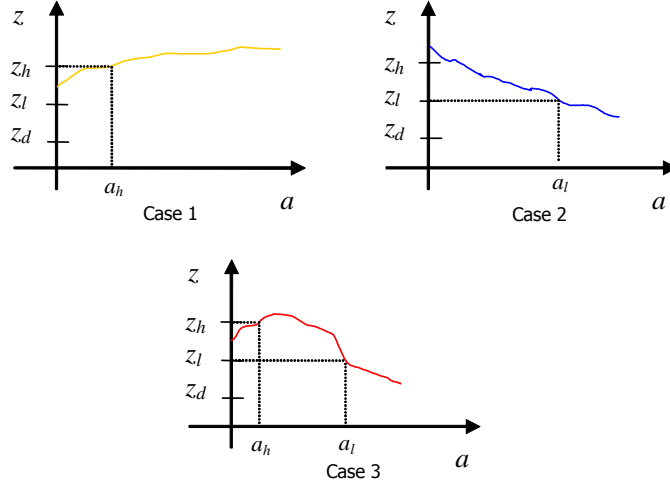


Figure 5: Examples of Feasible Productivity and Exporting Status Paths

Trade Equilibrium

To determine the equilibrium under the assumption of positive reentry cost we proceed as in the previous case. We only need to take into account the fact that, over the band of inaction, firms with the same productivity level can have different export status. In other words, we need to extend Luttmer's framework to determine the density of exporters $p_v(z)$ over the band of inaction. Define the probability that a firm has productivity z and is exporting at age a as $v(a, z)$ and the probability that a firm has productivity z and is nonexporting at age a as $w(a, z) = u(a, z) - v(a, z)$.

Then $v(a, z)$ and $w(a, z)$ jointly satisfy

$$\begin{cases} \frac{\partial v(a, z)}{\partial a} = -\alpha \frac{\partial v(a, z)}{\partial z} + \frac{1}{2} \xi^2 \frac{\partial^2 v(a, z)}{\partial z^2} + J_w(a, z_h^-) \delta(z - z_h) & z_l < z < z_h \\ v(a, z) = 0 \text{ for } z \leq z_l, a \geq 0 ; v(0, z) = \delta(z - \bar{z}) \end{cases}$$

and

$$\begin{cases} \frac{\partial w(a, z)}{\partial a} = -\alpha \frac{\partial w(a, z)}{\partial z} + \frac{1}{2} \xi^2 \frac{\partial^2 w(a, z)}{\partial z^2} - J_v(a, z_l^+) \delta(z - z_l) & z_l < z < z_h \\ w(a, z) = 0 \text{ for } z \geq z_h, a \geq 0 ; w(0, z) = \delta(z - \bar{z}) \end{cases}$$

where the additional (with respect to (24)) J_w and J_v terms account for the influx of newly exporting firms and newly nonexporting firms respectively. In the Appendix

we show how to derive the solutions of these linked partial differential equations. It turns out that,

$$\begin{aligned} v(z, a|\bar{z}) &= u(a, z - (z_l - z_d)|\bar{z} - (z_l - z_d))1_{\{\bar{z} > z_h\}} \\ &\quad - \gamma_v u(a, z - (z_l - z_d)|z_h - (z_l - z_d)), \end{aligned}$$

where γ_v is a constant term that depends on z_h and \bar{z} . The density of exporters over the band of inaction $p_v(z)$ can be determined in a way similar to how we determined the overall density $p_u(z)$,

$$p_v(z) = \frac{\int_{z_h}^{\infty} v(z|\bar{z}) \left[\frac{1 - e^{-\theta_*(\bar{z} - z_l)} - \gamma_{pv}(1 - e^{-\theta_*(z_h - z_l)})}{\kappa} \right] g_u(\bar{z}) d\bar{z}}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z} - z_l)} - \Upsilon(1 - e^{-\theta_*(z_h - z_l)})}{\kappa} \right] g_u(\bar{z}) d\bar{z} \right]}.$$

We can therefore determine the equilibrium price index in the economy using the expressions for the productivity cutoffs z_d , z_l and z_h together with,

$$P^{1-\sigma} = M \int_{z_d}^{\infty} \left(\frac{\bar{m}}{e^z} \right)^{1-\sigma} p_u(z) dz + (N-1)M \left[\int_{z_l}^{z_h} \left(\frac{\tau \bar{m}}{e^z} \right)^{1-\sigma} p_v(z) dz + \int_{z_h}^{\infty} \left(\frac{\tau \bar{m}}{e^z} \right)^{1-\sigma} p_u(z) dz \right]$$

As before new firms enter the industry until the expected value of the firm net of entry cost is zero, that is

$$f_e = \int_{z_d}^{z_h} [V_d(\bar{z}) + (N-1)V_l(\bar{z})] dG(\bar{z}) + (N-1) \int_{z_h}^{+\infty} [V_h(\bar{z}) - V_l(\bar{z})] dG(\bar{z}) \quad (23)$$

where the equilibrium condition for the cutoffs are given by (10) and by the system (22). The remaining steps are similar to those explained earlier for the no-cost of entry case. The labor market equations are

$$\begin{aligned} L_e &= f_e M_a, \\ L_f &= M \left[f_d + f_x \int_{z_l}^{\infty} p_v(z) dz \right], \\ L_p &= MRP^{\sigma-1} \bar{m}^{-\sigma} \left[\int_{z_d}^{\infty} (e^z)^{\sigma-1} p_u(z) dz + (N-1) \tau^{-\sigma} \int_{z_l}^{\infty} (e^z)^{\sigma-1} p_v(z) dz \right], \\ L_h &= f_h M_a \left[(1 - G(z_h)) - \frac{1}{2} \frac{\partial \int_0^{\infty} \int w(a, z|\bar{z}) g(\bar{z}) e^{-\kappa a} d\bar{z} da}{\partial z} \Big|_{z=z_h} \right], \end{aligned}$$

where the L_h captures the amount of labor used to finance the sunk cost of exporting. This cost is sustained by new firms that are born as exporters (the first term in brackets is the probability of this event) and by incumbents that start exporting (the second term in brackets is the influx of firms that switch export status).

We just need to formally define the equilibrium concept for this case. An open-economy, entry-cost, stationary equilibrium is defined by

$$\left[c^h(u) \right]_{u \in \Omega, h \in \Omega_h}, \left\{ \left[p^i(z_a) \right]_{a \geq 0}, \left[l^i(z_a) \right]_{a \geq 0}, d_{a \geq 0}^i, e_{a \geq 0}^i \right\}_{i \in \Omega_I}$$

where (i) each consumer h chooses, in each period, from the set of available varieties Ω , to consume $c(u)$ units of good u to maximize the intertemporal utility function of equation (1) subject to an intertemporal budget constraint; (ii) incumbent firms choose, at every age a , price $p(z_a)$ and variable labor $l(z_a)$ in order to maximize domestic profits, given by equation (7), and eventual exporting profits, given by equation (17), taking the price index of the economy as given; (iii) incumbent firms, at every age a , decides if to keep being active ($d_a = 1$) or to exit ($d_a = 0$) and if to export ($e_a = 1$) or not ($e_a = 0$) knowing that productivity evolves according to the Brownian motion of equation (4) and that entry into the export market requires the payment of a sunk cost f_h ; (iv) firms enter, if the productivity draw is bigger than z_d , until the free-entry condition (23) is satisfied; (v) labor and goods markets clear.

6 Simulation

In this section we simulate the model, starting with the analysis of the open economy version without export entry cost. Our initial purpose is to show that the model is capable of generating relationships that are consistent with those seen in the data. This analysis is preparatory to the calibration of the model which is the subject of ongoing research.

We start by deriving the stationary equilibrium for the case of two countries with a labor force in the manufacturing sector comparable to the one of Chile (about

Table 6: Parameters Values

Parameter	f_e	f_d	f_x	α	ξ
Value	.0005	.1	.2	.03	.1

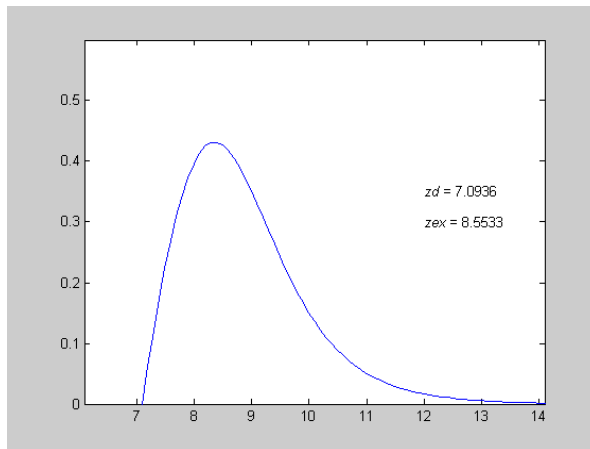


Figure 6: Equilibrium Stationary Productivity Density, $\tau = 1.25$

420,000 workers), a long-term interest rate of 6%, an elasticity of substitution equal to 2 (slightly lower than the one estimated by Irarrazabal and Oromolla (2005)) and an iceberg cost τ equal to 1.25. We assume that the initial log productivity density $g(\cdot)$ is Normal, like in the data, and has mean 7.7 and standard deviation .86 while the killing rate is 4%. The other parameters values are summarized in Table (6).¹²

Figure (6) shows the stationary productivity density. The lower productivity cutoff z_d is equal to about 7.09 while the export cutoff z_{ex} is about 8.55. This implies that the average productivity of exporters is higher than the one of nonexporters by a factor of 5 while the fraction of exporting firms is 55.5%.

Figure (7) shows the effects of a reduction in trade barriers . When the iceberg cost τ is reduced to 1.1 (dashed curve), both the lower productivity cutoff z_d and the export cutoff z_{ex} increase. As a consequence of this selection, average productivity

¹²The purpose of this section is to study the properties of the model. At this stage, not all the parameters values are consistent with what observed in the data.

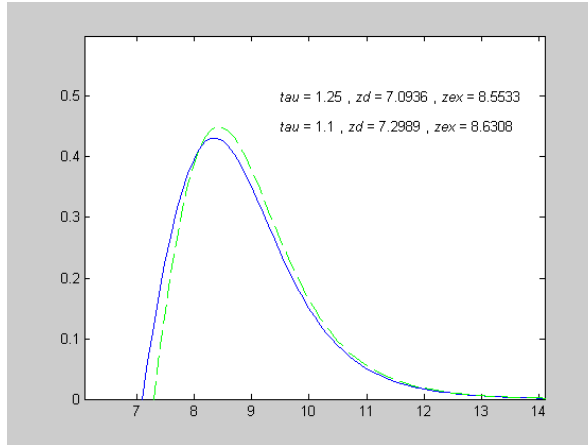


Figure 7: Reduction of Trade Barriers: from $\tau = 1.25$ to $\tau = 1.1$

increases by 11% and the price index decreases by 14%.

7 Conclusions and Extensions

This paper presents a model of international trade with heterogenous producers. Our contribution is to introduce firms specific productivity shocks and derive a stationary industry equilibrium in a multi-country general equilibrium framework.

We first embed firms subject to ex-post uncertainty into a monopolistic competition model. We derive the stationary distribution of firm characteristics and establish the conditions for the equilibrium of the economy under autarky. We then determine an equilibrium for an integrated world market with symmetric countries. Several results of previous trade models with heterogenous producers are derived. We then show how uncertainty alters in a nontrivial way some of the conclusions of previous studies. In particular, we show that introducing positive export entry costs in a framework where productivity is evolving stochastically changes the well know partition of firms by exporting status. We derive explicit conditions to determine the factors that affect the band of inaction in which domestic firms continue to sell domestically and

exporting firms continue to export. In this context, we derive the stationary density of exporters along the band of inaction, this derivation is potentially interesting of its own and could be usefully applied to other similar economic problems.

Our model retains the prediction that exporters are more productive than nonexporters but also provides one reason why some nonexporters are more efficient than some exporters. Additionally, in our framework, both entrants and incumbent firms might sustain temporary negative profits because of the expectation of becoming profitable later on. An important feature of the model is that it is amenable to simulation and estimation and can be used as an effective tool to better understand the consequences of trade opening and trade policies. Preliminary results on the simulation of the model in open economy are provided. We derived closed-form solutions for several static and dynamic moments that can be used to match the model to the data. The model can then be easily extended to analyze the effects of trade policies in a context of multiple asymmetric countries. The study of transition dynamics is a more complex but natural extension. Finally, we are interested in endogenizing the process for productivity change, maybe through the introduction of learning by exporting, and be able to provide new insights in the debate on the direction of the relationship between productivity and export participation. All these are subjects of ongoing research.

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A Appendix

A.1 Value of the Firm

A.1.1 Domestic Component

The value of a firm that is selling only on the domestic market $V_d(z)$ and the corresponding productivity cutoff z_d are the solutions of the following ordinary differential equation

$$\begin{aligned} -(r + \kappa)V_d(z) + \frac{1}{2}\xi^2 V_d''(z) + \alpha V_d'(z) + \pi_d(z) &= 0 \\ \text{s.t. } V_d(z_d) &= 0 \quad \text{and} \quad V_d'(z_d) = 0 \end{aligned}$$

where $\pi_d(z) = A_d e^{(\sigma-1)z} + B_d$ with $A_d = \frac{R}{\sigma} (\bar{m}^{-1}P)^{\sigma-1}$ and $B_d = -f_d$. The general solution of the differential equation is

$$V_d(z) = V_d^p(z) + V_d^h(z)$$

where $V_d^p(z)$ is a particular solution of the non-homogeneous ode and $V_d^h(z)$ is the general solution of the homogeneous ode. The latter has the form

$$V_d^h(z) = c_1 e^{\beta_1 z} + c_2 e^{\beta_2 z}$$

where c_1 and c_2 are two constants to be determined and $\beta_1 > 0$ and $\beta_2 < 0$ are the roots of the characteristic equation associated to the homogeneous ode.

The general solution of the homogeneous equation represents the value of the option to exit (as we are going to show later in this section). The likelihood of abandonment in the not-too-distant future becomes extremely small as z goes to ∞ , so the value of the abandonment option should go to zero as z becomes very large. Hence the coefficient c_1 corresponding to the positive root β_1 should be zero. This leaves

$$V_d^h(z) = c_2 e^{\beta_2 z}$$

We need to find the particular solution to the non-homogeneous ode. Using the "undetermined coefficients" method, the particular solution when the forcing term

has the form $A_d e^{(\sigma-1)z} + B_d$ is $V_d^p(z) = C e^{(\sigma-1)z} + D$. Hence, we just need to plug this into the non-homogeneous ode and find C and D that makes $V_d^p(z)$ a particular solution. This delivers

$$V_d^p(z) = -\frac{A_d}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z} + \frac{B_d}{(r+\kappa)}$$

and

$$V_d(z) = c_2 e^{\beta_2 z} - \frac{A_d}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z} + \frac{B_d}{(r+\kappa)}$$

Now using the boundary conditions $V_d(z_d) = 0$ and $dV_d(z_d)/dz = 0$ we can determine c_2 and in particular z_d ,

$$e^{z_d} = \frac{\bar{m}}{P} \left(\frac{f_d \sigma}{R} \right)^{\frac{1}{(\sigma-1)}} \gamma_d$$

where $\gamma_d = \left[\frac{-\beta_2}{(r+\kappa)} \frac{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)}{\beta_2 - (\sigma-1)} \right]^{\frac{1}{(\sigma-1)}}$.

We can now express the general solution to the ode as a function of z_d ,

$$V_d(z) = \frac{f_d}{(r+\kappa)} \frac{\beta_2}{\beta_2 - (\sigma-1)} \left[e^{(\sigma-1)(z-z_d)} - 1 - \frac{(\sigma-1)}{\beta_2} \left(e^{\beta_2(z-z_d)} - 1 \right) \right]$$

Note that

$$V_d'(z) = \frac{f_d}{(r+\kappa)} \frac{\beta_2(\sigma-1)}{\beta_2 - (\sigma-1)} \left[e^{(\sigma-1)(z-z_d)} - e^{\beta_2(z-z_d)} \right] > 0$$

since $\sigma > 1$ and $\beta_2 < 0$.

Interpretation of $V_d(z)$

1. The two components, $V_d^p(z)$ and $V_d^h(z)$, of the general solution of the ode have a straightforward interpretation. Ito's Lemma and the stochastic process for z imply that the stochastic process for the domestic variable profits of a firm $\pi_d^v(z) = \frac{R}{\sigma} (\bar{m}^{-1} P e^z)^{\sigma-1}$ is a geometric Brownian motion with drift

$[\alpha(\sigma - 1) + 1/2\xi^2(\sigma - 1)^2] \pi_d^v(z)$ and diffusion coefficient $\xi(\sigma - 1)\pi_d^v(z)$. Denoting today's variable profits by $\pi_d^v(z_a)$, the expected value and variance of variable profits a^* years from now are

$$\begin{aligned} E[\pi_{d,a+a^*}^v] &= \pi_d^v(z_a) e^{[\alpha(\sigma-1)+1/2\xi^2(\sigma-1)^2]a^*} \\ V[\pi_{d,a+a^*}^v] &= \pi_d^v(z_a) e^{2[\alpha(\sigma-1)+1/2\xi^2(\sigma-1)^2]a^*} (e^{\xi^2(\sigma-1)^2 a^*} - 1) \end{aligned}$$

so that the expected present discounted value of variable profits over an infinite period of time is

$$E \left[\int_0^\infty \pi_{d,a+a^*}^v e^{-(r+\kappa)a^*} da^* \right] = \frac{\pi_d^v(z_a)}{(r + \kappa) - \alpha(\sigma - 1) - 1/2\xi^2(\sigma - 1)^2}$$

which represents the value of a firm without fixed costs f_d . Since f_d is constant over time, the expected present discounted value of total profits over some period of time is

$$\frac{\pi_d^v(z_a)}{(r + \kappa) - \alpha(\sigma - 1) - 1/2\xi^2(\sigma - 1)^2} - \frac{f_d}{(r + \kappa)} = V_d^p(z)$$

so that the other component of the general solution of the ode, $V_d^h(z)$, represents the value of the option to exit. Note that $V_d^h(z) > 0$ since $c_2 > 0$.

2. The exit cutoff z_d is smaller than the zero-profit cutoff z_0 . Note that $e^{z_0} - e^{z_d} = (1 - \gamma_d)e^{z_0}$. So we just need to prove that $\gamma_d < 1$. This condition can be expressed as

$$(r + \kappa) > \alpha\beta_2 + 1/2\xi^2(\sigma - 1)\beta_2$$

which is satisfied when the drift is positive, $\alpha > 0$, since $\beta_2 < 0$ and $\sigma > 1$.

A.1.2 Export Component

The value of the firm due to export is the solution to the following ordinary differential equation

$$-(r + \kappa)V_{ex}(z) + \frac{1}{2}\xi^2 V_{ex}''(z) + \alpha V_{ex}'(z) + \pi_{ex}(z) = 0$$

where $\pi_{ex}(z) = \max [A_{ex}e^{(\sigma-1)z} + B_{ex}, 0]$ with $A_{ex} = \tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1}P)^{\sigma-1}$ and $B_{ex} = -f_x$. Since the forcing function is defined differently when current export profits are positive or negative we need to solve the equation separately for the two cases and then stitch together the two solutions at the point where $\pi_{ex}(z) = 0$, that is at

$$e^{z_{ex}} = \frac{\tau \bar{m}}{P} \left(\frac{\sigma f_x}{R} \right)^{\frac{1}{\sigma-1}}$$

In the region $z < z_{ex}$, we have that $\pi_{ex}(z) = 0$ and only the homogeneous part of the equation remains. Therefore the general solution is

$$V_{ex}^{h<}(z) = k_1 e^{\beta_1 z} + k_2 e^{\beta_2 z}$$

where k_1 and k_2 are two constants to be determined and β_1 and β_2 are the roots of the characteristic equation associated to the homogeneous ode.

In the region $z > z_{ex}$, we take another linear combination of the exponential solutions of the homogeneous part, and add on any particular solution of the full equation. Using the "undetermined coefficients" method, the particular solution is

$$V_{ex}^p(z) = -\frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z} + \frac{B_{ex}}{(r+\kappa)}$$

and the general solution for the case $z > z_{ex}$ is

$$V_{ex}^{h>}(z) = b_1 e^{\beta_1 z} + b_2 e^{\beta_2 z} - \frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z} + \frac{B_{ex}}{(r+\kappa)}$$

Now note that, in the region $z < z_{ex}$, operation is suspended but there is a positive probability that the productivity process will at some future time move into the region $z > z_{ex}$, when operation will resume and profits from exporting accrue. The value $V_{ex}^{h<}(z)$ when $z < z_{ex}$, is just the expected present value of such future flows. As z becomes very small the event of its rising above z_{ex} is very unlikely and so the value $V_{ex}^{h<}(z)$ should go to zero. We can therefore set the constant k_2 , corresponding to the negative root β_2 , to zero.

Now let's consider the region $z > z_{ex}$. The part of $V_{ex}^{h>}(z)$ different from the particular solution represents the additional value of the option to suspend operations in the future should z fall below z_{ex} . As z becomes very large the value of this option should go to zero and so we can set to zero b_1 , the constant associated to the positive root β_1 . We have then

$$V_{ex}(z) = \begin{cases} k_1 e^{\beta_1 z} & z < z_{ex} \\ b_2 e^{\beta_2 z} - \frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z} + \frac{B_{ex}}{(r+\kappa)} & z > z_{ex} \end{cases}$$

Note that since the Brownian motion can diffuse freely across the z_{ex} boundary, the value function cannot change abruptly across it. The solution must be continuously differentiable across z_{ex} . We therefore have the following two conditions

$$\begin{aligned} k_1 e^{\beta_1 z_{ex}} &= b_2 e^{\beta_2 z_{ex}} - \frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z_{ex}} + \frac{B_{ex}}{(r+\kappa)} \\ k_1 \beta_1 e^{\beta_1 z_{ex}} &= b_2 \beta_2 e^{\beta_2 z_{ex}} - \frac{A_{ex}(\sigma-1)}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z_{ex}} \end{aligned}$$

which is a system of two equations in two unknowns, k_1 and b_2 . Given k_1 and b_2 (both nonnegative), we can find the final expression for $V_{ex}(z)$. For the region where $z > z_{ex}$ we have equation (19),

$$\begin{aligned} V_{ex}(z) &= \frac{f_x}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} \left[e^{\beta_2(z-z_{ex})} \left(\frac{\sigma-1-\beta_1}{\beta_2-\beta_1\beta_2} \right) - e^{(\sigma-1)(z-z_{ex})} \right] \\ &\quad - \frac{f_x}{(r+\kappa)} \left[\frac{\beta_1}{\beta_2-\beta_1\beta_2} e^{\beta_2(z-z_{ex})} + 1 \right] \end{aligned}$$

while for the region where $z < z_{ex}$ we have equation (20),

$$V_{ex}(z) = \left\{ \frac{f_x}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} \frac{\beta_2(\sigma-2)}{\beta_1} - \beta_2 \frac{f_x}{(r+\kappa)} \right\} \frac{e^{\beta_1(z-z_{ex})}}{\beta_2 - \beta_1\beta_2}$$

A.2 Stationary Distribution

A.2.1 The Transition Density $u(a, z|\bar{z})$

The solution of the Kolmogorov equation for firms with initial productivity \bar{z} is given by,

$$u(a, z|\bar{z}) = \frac{1}{\xi\sqrt{a}} \left[\phi\left(\frac{z - \bar{z} - \alpha a}{\xi\sqrt{a}}\right) - e^{-\frac{2\alpha}{\xi^2}(\bar{z}-z_d)} \phi\left(\frac{z + \bar{z} - 2z_d - \alpha a}{\xi\sqrt{a}}\right) \right]$$

the transition density of a Brownian Motion (BM) like the one of equation (4), subject to a lower absorbing barrier $z_d < \bar{z}$. Karlin and Taylor (1998) derive the transition density for a BM without drift, with diffusion coefficient ξ equal to one and lower barrier z_d equal to zero:

$$\Pr(z_a > y|\bar{z}) = \Phi\left(\frac{y + \bar{z}}{\sqrt{a}}\right) - \Phi\left(\frac{y - \bar{z}}{\sqrt{a}}\right)$$

The corresponding solution for the case where $\xi > 0$ and $z_d \neq 0$ can be easily, using the reflection principle, shown to be

$$\Pr(z_a > y|\bar{z}) = \Phi\left(\frac{y + \bar{z} - 2z_d}{\xi\sqrt{a}}\right) - \Phi\left(\frac{y - \bar{z}}{\xi\sqrt{a}}\right)$$

In order to derive $u(a, z|\bar{z})$, just apply the Change of Measure Theorem (see Harrison (1985)) using the following Radon-Nikodym derivative

$$dP^* = \chi dP$$

where

$$\chi = \exp\left[\frac{\alpha}{\xi^2}(z - \bar{z}) - \frac{1}{2}\frac{\alpha^2}{\xi^2}a\right]$$

A.2.2 An Alternative Derivation for Both $u(a, z|\bar{z})$ and $v(a, z|\bar{z})$

We describe here a more rigorous way to derive $u(a, z|\bar{z})$. This alternative proof is going to be important because it allows to easily derive the density of exporters in the hysteresis case. Let $u(a, z)$ be the probability that a firm has productivity z at age a . Let then $v(a, z)$ be the probability that a firm has productivity z and is exporting at age a . The probability that a firm has productivity z and is nonexporting at age a is $w(a, z) = u(a, z) - v(a, z)$. Then $u(\cdot)$ satisfies the following system,

$$\begin{cases} \frac{\partial u(a, z)}{\partial a} = -\alpha \frac{\partial u(a, z)}{\partial z} + \frac{1}{2}\xi^2 \frac{\partial^2 u(a, z)}{\partial z^2} & z > z_d \\ u(a, z_d) = 0 \text{ for } a \geq 0 ; u(0, z) = g(z)1_{\{z > z_d\}} = g_u(z) \end{cases} \quad (24)$$

while $v(a, z)$ and $w(a, z)$ jointly satisfy

$$\begin{cases} \frac{\partial v(a, z)}{\partial a} = -\alpha \frac{\partial v(a, z)}{\partial z} + \frac{1}{2}\xi^2 \frac{\partial^2 v(a, z)}{\partial z^2} + J_w(a, z_h^-)\delta(z - z_h) & z_l < z < z_h \\ v(a, z) = 0 \text{ for } z \leq z_l, a \geq 0 ; v(0, z) = g(z)1_{\{z > z_h\}} = g_v(z) \end{cases} \quad (25)$$

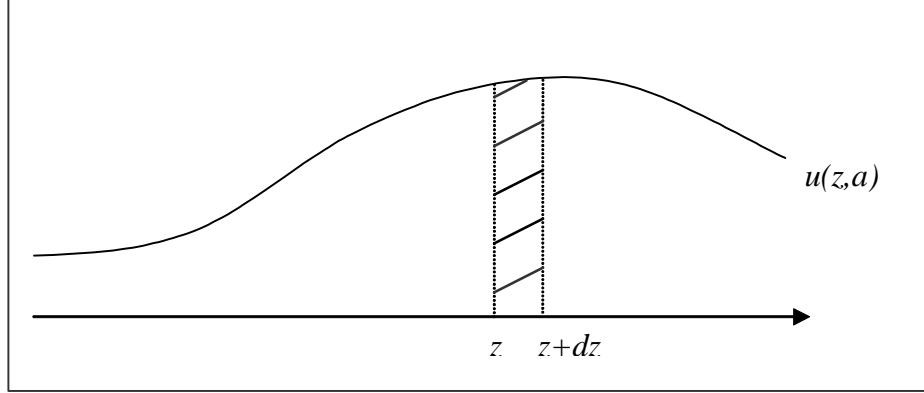


Figure 8:

and

$$\begin{cases} \frac{\partial w(a, z)}{\partial a} = -\alpha \frac{\partial w(a, z)}{\partial z} + \frac{1}{2} \xi^2 \frac{\partial^2 w(a, z)}{\partial z^2} - J_v(a, z_l^+) \delta(z - z_l) & z_l < z < z_h \\ w(a, z) = 0 \text{ for } z \geq z_h, a \geq 0; w(0, z) = g(z) 1_{\{z_d < z < z_h\}} = g_w(z) \end{cases} \quad (26)$$

The difference between (24) and (25) is the term $J_w(a, z_h^-) \delta(z - z_h)$ where $\delta(\cdot)$ is the Dirac's delta function. This term captures the influx of newly exporting firms, that is, firms that were not exporting but whose productivity overtakes the threshold z_h and therefore start exporting. Since this is the main difference in the equations for u and v we are going to provide the intuition underlying the flux term. Let's go back to $u(a, z)$ for a moment. Figure (8) plots $u(a, z)$ against z for a particular age a . Let's define by $J_u(a, z)$ the net rate of passage or flux of particles at z (in the z direction). Consider the change in the probability mass in the shaded area when age changes infinitesimally. This is equal to $\frac{\partial u(a, z) dz}{\partial a} = J_u(a, z) - J_u(a, z + dz)$ so that $\frac{\partial u(a, z)}{\partial a} = -\frac{\partial J_u(a, z)}{\partial z}$. Note that the probability mass in the shaded area increases when $\frac{\partial J_u(a, z)}{\partial z} < 0$ that is when the mass of particles exiting from the shaded area is bigger than the mass of entering particles. Going back to (24), we can see that $J_u(a, z) = \alpha u(a, z) - \frac{1}{2} \xi^2 \frac{\partial u(a, z)}{\partial z}$. Therefore we have

$$\int_z^{z+dz} \frac{\partial u(a, s)}{\partial a} ds = \frac{\partial}{\partial a} \int_z^{z+dz} u(a, s) ds = -[J_u(a, z + dz) - J_u(a, z)] \quad z > z_d$$

Now we can consider the pde for $v(a, z)$ and conclude that

$$\frac{\partial}{\partial a} \int_z^{z+dz} v(a, s) ds = -[J_v(a, z + dz) - J_v(a, z)] + \int_z^{z+dz} J_w(a, z_h^-) \delta(s - z_h) ds \quad z > z_d$$

so that when $z = z_h$ we have,

$$\frac{\partial}{\partial a} \int_{z_h}^{z_h+dz} v(a, s) ds = -[J_v(a, z_h + dz) - J_v(a, z_h)] + J_w(a, z_h^-)$$

which shows that the change in the mass of exporting firms with productivity slightly higher than z_h depends on the mass of exporting firms whose productivity becomes slightly higher than z_h and on the mass of newly exporting firms. A similar intuition is behind the presence of the $J_v(z_l^+, a) \delta(z - z_l)$ term in (26).

Now we can solve (24),(25) and (26) by using Laplace transforms of $u()$, $v()$ and $w()$. First, note that since $w(a, z_h) = v(a, z_l) = 0$ the flux terms simplify to $J_w(a, z_h^-) = -\frac{1}{2}\xi^2 \frac{\partial w(a, z_h^-)}{\partial z}$ and $J_v(a, z_l^+) = -\frac{1}{2}\xi^2 \frac{\partial v(a, z_l^+)}{\partial z}$. Let $\hat{u}(z) = \int_0^\infty e^{-\chi a} u(a, z) da$ be the Laplace transform of $u(a, z)$.¹³ The system (24) can be rewritten as

$$\begin{cases} \frac{1}{2}\xi^2 \hat{u}''(z) - \alpha \hat{u}'(z) - \chi \hat{u}(z) = -g_u(z) & z > z_d \\ \hat{u}(z_d) = 0 \end{cases} \quad (27)$$

To solve this ode we need first the solutions of the homogeneous equation (i.e. $g(z) = 0$). Let $h_1(z) = e^{\lambda_1 z}$ and $h_2(z) = e^{\lambda_2 z}$ where $\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 2\xi^2 \chi}}{\xi^2}$ be the solutions of the homogeneous equation and let $\lambda_1 < 0$ and $\lambda_2 > 0$. Let $h_-(.)$ and $h_+(.)$ be two linear combinations of $h_1(.)$ and $h_2(.)$ such that $\lim_{z \rightarrow \infty} h_+(z) = 0$ and $h_-(z_d) = 0$. The following claim describes the general solution of the nonhomogeneous equation:

Claim 3 $\hat{u}(z) = \int_{-\infty}^\infty S(z, y) g_u(y) \Delta(y) dy$ satisfies (27) where

$$\Delta(y) = \left[\frac{1}{2}\xi^2 (h'_-(y)h_+(y) - h_-(y)h'_+(y)) \right]^{-1} \text{ and } S(z, y) = \begin{cases} h_+(z)h_-(y) & \text{for } y < z \\ h_-(z)h_+(y) & \text{for } y > z \end{cases}$$

¹³We suppress the χ argument in order to simplify the notation.

Let $h_+(z) = e^{\lambda_1 z}$ and $h_-(z) = e^{\lambda_1 z} + B e^{\lambda_2 z}$. Using the boundary condition $h_-(z_d) = 0$ we can find the constant $B = -e^{(\lambda_1 - \lambda_2)z_d}$ so that $h_-(z) = e^{\lambda_1 z} - e^{\lambda_1 z_d} e^{\lambda_2(z - z_d)}$. Using the fact that $h_+(\cdot)$ and $h_-(\cdot)$ are linear combinations of the solutions of the homogeneous equation we can find $\Delta(y) = \frac{2}{\xi^2(\lambda_1 - \lambda_2)} e^{(\lambda_2 - \lambda_1)z_d} e^{-\frac{2\alpha}{\xi^2}y}$.

Let $K(z, y) = S(z, y)\Delta(y)$. The solution to (27) is then

$$\hat{u}(z) = \int_{-\infty}^{\infty} K(z, y)g_u(y)dy$$

The inverse Laplace transform of $\hat{u}(z)$ is the solution to (24). In the case $g_u(\bar{z}) = \delta(z - \bar{z})$, $\hat{u}(z) = K(z, \bar{z})$ and therefore the inverse Laplace transform of $\hat{u}(z)$ is $u(a, z|\bar{z})$ which, as we showed in the previous section, is equal to (??).

Let's now solve (25). Let $\hat{v}(z) = \int_0^{\infty} e^{-\chi a} v(a, z) da$ be the Laplace transform of $v(a, z)$. Similarly to what we did earlier we can rewrite the systems for $v(a, z)$ as

$$\begin{cases} \frac{1}{2}\xi^2 \hat{v}''(z) - \alpha \hat{v}'(z) - \chi \hat{v}(z) = -g_v(z) + \frac{1}{2}\xi^2 \hat{w}'(z_h) \delta(z - z_h) & z > z_l \\ \hat{v}(z_l) = 0 \end{cases} \quad (28)$$

The solution to this ode is similar to the previous one. Define $K_v(z, y) = K(z - (z_l - z_d), y - (z_l - z_d))$. Then the solution to (28) is

$$\begin{aligned} \hat{v}(z) &= \int_{z_l}^{\infty} K_v(z, y) \left[g_v(y) - \frac{1}{2}\xi^2 \hat{w}'(z_h) \delta(z - z_h) \right] dy \\ &= \int_{z_h}^{\infty} K_v(z, y) g_v(y) dy - \frac{1}{2}\xi^2 \hat{w}'(z_h) K_v(z, z_h) \end{aligned}$$

The inverse Laplace transform of $\hat{v}(z)$ is the solution to (25), $v(a, z)$. For the case $g_v(\bar{z}) = \delta(z - \bar{z})$, the solution depends on the particular value of \bar{z} ,

$$\hat{v}(z) = K_v(z, \bar{z}) \mathbf{1}_{\{\bar{z} > z_h\}} - \frac{1}{2}\xi^2 \hat{w}'(z_h) K_v(z, z_h)$$

Note that

$$\hat{v}'(z_h^-) = \frac{\partial}{\partial z} \Big|_{z=z_h^-} K_v(z, \bar{z}) \mathbf{1}_{\{\bar{z} > z_h\}} - \frac{1}{2}\xi^2 [\hat{u}'(z_h^-) - \hat{v}'(z_h^-)] \frac{\partial}{\partial z} \Big|_{z=z_h^-} K_v(z, z_h)$$

so that

$$\hat{v}(z) = K_v(z, \bar{z})1_{\{\bar{z} > z_h\}} - \frac{1}{2}\xi^2 \frac{K'(z_h^-, \bar{z}) - K'_v(z_h^-, \bar{z})1_{\{\bar{z} > z_h\}}}{1 - \frac{\xi^2}{2}K'_v(z_h^-, z_h)}K_v(z, z_h)$$

Since we already know that $v(a, z|\bar{z})$ is the inverse Laplace transform of $K(z, \bar{z})$ and that $K_v(z, y) = K(z - (z_l - z_d), y - (z_l - z_d))$, it's easy to find the inverse Laplace transform of $\hat{v}(z)$,

$$\begin{aligned} v(a, z|\bar{z}) &= u(a, z - (z_l - z_d)|\bar{z} - (z_l - z_d))1_{\{\bar{z} > z_h\}} \\ &\quad - \frac{1}{2}\xi^2 \frac{K'(z_h^-, \bar{z}) - K'_v(z_h^-, \bar{z})1_{\{\bar{z} > z_h\}}}{1 - \frac{\xi^2}{2}K'_v(z_h^-, z_h)}u(a, z - (z_l - z_d)|z_h - (z_l - z_d)) \end{aligned}$$

A.2.3 The Probability Densities $p_u(z|\bar{z})$ and $p_u(a, z|\bar{z})$

We start by modifying $u(a, z|\bar{z})$ in order to take into account the killing rate κ ,

$$\check{u}(a, z|\bar{z}) = e^{-\kappa a}u(a, z|\bar{z})$$

Integrating out a we obtain an expression for the marginal probability of productivities given an initial starting level. We are going to consider only $\bar{z} > z_d$, so that, since $z > z_d$, we have $z + \bar{z} - 2z_d > 0$.

$$\begin{aligned} \check{u}(z|\bar{z}) &= \int_0^\infty \check{u}(a, z|\bar{z})da \\ &= \int_0^\infty e^{-\kappa a} \frac{1}{\xi\sqrt{a}} \phi\left(\frac{z - \bar{z} - \alpha a}{\xi\sqrt{a}}\right) da - e^{-\frac{2\alpha}{\xi^2}(\bar{z} - z_d)} \int_0^\infty e^{-\kappa a} \phi\left(\frac{z + \bar{z} - 2z_d - \alpha a}{\xi\sqrt{a}}\right) da \\ &= m_1 - e^{-\frac{2\alpha}{\xi^2}(\bar{z} - z_d)} m_2 \end{aligned}$$

where the solutions of the two integrals m_1 and m_2 are,

$$\begin{aligned} m_1 &= \frac{\min\{e^{\theta_*(z - \bar{z})}, e^{-\theta(z - \bar{z})}\}}{\sqrt{\alpha^2 + 2\xi^2\kappa}} \\ m_2 &= \frac{\exp(-\theta[z + \bar{z} - 2z_d])}{\sqrt{\alpha^2 + 2\xi^2\kappa}} \end{aligned}$$

and θ and θ_* are two non-negative constants such that

$$\begin{aligned} \theta &= \frac{1}{\xi^2} \left(-\alpha + \sqrt{\alpha^2 + 2\xi^2\kappa} \right) \\ \theta_* &= \frac{1}{\xi^2} \left(\alpha + \sqrt{\alpha^2 + 2\xi^2\kappa} \right) \end{aligned}$$

Now we can combine the results to get

$$\check{u}(z|\bar{z}) = -\frac{1}{\alpha} \frac{\theta - \theta_*}{\theta + \theta_*} \frac{[\min \{e^{(\theta+\theta_*)(z-z_d)}, e^{(\theta+\theta_*)(\bar{z}-z_d)}\} - 1]}{e^{\theta_*(\bar{z}-z_d)} e^{\theta(z-z_d)}}$$

The last step is to convert $\check{u}(z|\bar{z})$ into a probability density. Integrating $\check{u}(z|\bar{z})$ over z , we find

$$\int_{z_d}^{\infty} \check{u}(z|\bar{z}) dz = -\frac{1}{\alpha} \frac{\theta - \theta_*}{\theta + \theta_*} [1 - e^{-\theta_*(\bar{z}-z_d)}] = \frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\kappa}$$

that can be used to transform $\check{u}(z|\bar{z})$ and $\check{u}(a, z|\bar{z})$ into the probability densities (13)

$$p_u(z|\bar{z}) = \frac{\check{u}(z|\bar{z})}{\int_{z_d}^{\infty} \check{u}(z|\bar{z}) dz} = \frac{\theta}{\theta + \theta_*} \left[\frac{\min \{e^{(\theta_*)+(z-z_d)}, e^{(\theta_*)+(\bar{z}-z_d)}\} - 1}{e^{\theta(z-z_d)} \left(\frac{e^{\theta_*(\bar{z}-z_d)} - 1}{\theta_*} \right)} \right]$$

and

$$p_u(a, z|\bar{z}) = \frac{\check{u}(a, z|\bar{z})}{\int_{z_d}^{\infty} \check{u}(z|\bar{z}) dz} = \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\kappa} \right]^{-1} e^{-\kappa a} u(a, z|\bar{z})$$

A.2.4 Mean Value of Productivity

We need to calculate the mean of the stationary distribution as

$$\begin{aligned} e^{\bar{z}} &= \int_{z_d}^{\infty} e^{z-z_d} p_u(z|\bar{z}) dz \\ &= \int_{z_d}^{\infty} e^{z-z_d} \frac{\theta}{\theta + \theta_*} \left[\frac{\min \{e^{(\theta_*)+(z-z_d)}, e^{(\theta_*)+(\bar{z}-z_d)}\} - 1}{e^{\theta(z-z_d)} \left(\frac{e^{\theta_*(\bar{z}-z_d)} - 1}{\theta_*} \right)} \right] dz \\ &= \frac{\theta \theta_*}{\theta + \theta_*} \frac{1}{e^{\theta_*(\bar{z}-z_d)} - 1} \left\{ \begin{aligned} &\int_{z_d}^{\bar{z}} [e^{(\theta_*)+(z-z_d)} - e^{(1-\theta)(z-z_d)}] dz \\ &+ (e^{(\theta_*)+(\bar{z}-z_d)} - 1) e^{(1-\theta)(-z_d)} \int_{\bar{z}}^{\infty} e^{(1-\theta)z} dz \end{aligned} \right\} \end{aligned}$$

Assumption 2 guarantees that $\theta > 1$ which is necessary for the last integral, in the previous line, to be finite. We can then go on with,

$$\begin{aligned}
&= \frac{\theta\theta_*}{\theta + \theta_*} \frac{1}{e^{\theta_*(\bar{z}-z_d)} - 1} \left\{ \int_{z_d}^{\bar{z}} [e^{(\theta_*+1)(z-z_d)} - e^{(1-\theta)(z-z_d)}] dz \right. \\
&\quad \left. + (e^{(\theta+\theta_*)(\bar{z}-z_d)} - 1) e^{(1-\theta)(-z_d)} \frac{e^{(1-\theta)\bar{z}}}{(\theta-1)} \right\} \\
&= \frac{\theta}{(\theta-1)} \left[\frac{\frac{1-e^{-(1+\theta_*)(\bar{z}-z_d)}}{(\theta_*+1)}}{\frac{1-e^{-\theta_*(\bar{z}-z_d)}}{\theta_*}} \right] e^{\bar{z}-z_d}
\end{aligned}$$

Hence, we derive equation (14),

$$e^{\tilde{z}} = \frac{\theta}{(\theta-1)} \left[\frac{\frac{1-e^{-(1+\theta_*)(\bar{z}-z_d)}}{(\theta_*+1)}}{\frac{1-e^{-\theta_*(\bar{z}-z_d)}}{\theta_*}} \right] e^{\bar{z}-z_d}$$

A.2.5 Equilibrium Rate of Entry

Consider the stationary distribution

$$p_u(a, z, \bar{z}) = e^{-\kappa a} u(a, z|\bar{z}) g(\bar{z}) \frac{M_a}{M}$$

The latter is a probability density for a particular value of M_a/M . This is used to determine the amount of entry that must take place relative to the number of existing firms. We will use the fact that $p_u(a, z|\bar{z})$ is a probability density function.

$$1 = \int_{z_d}^{\infty} \int_{z_d}^{\infty} \int_0^{\infty} p_u(a, z, \bar{z}) da dz d\bar{z} = \int_{z_d}^{\infty} \int_{z_d}^{\infty} \int_0^{\infty} e^{-\kappa a} u(a, z|\bar{z}) g(\bar{z}) \frac{M_a}{M} da dz d\bar{z}$$

Replace the definition of $p_u(a, z|\bar{z})$,

$$1 = \int_{z_d}^{\infty} \int_{z_d}^{\infty} \int_0^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\kappa} \right] p_u(a, z|\bar{z}) g(\bar{z}) \frac{M_a}{M} da dz d\bar{z}$$

so that the equilibrium rate of entry is

$$\frac{M_a}{M} = \left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\kappa} \right] g(\bar{z}) d\bar{z} \right]^{-1}$$

A.2.6 The Stationary Distribution of Productivity

Use the attempted entry rate equation (15) to get the probability density

$$p_u(a, z, \bar{z}) = \frac{e^{-\kappa a} u(a, z | \bar{z}) g(\bar{z})}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z} - z_d)}}{\kappa} \right] g(\bar{z}) d\bar{z} \right]}$$

and replace the equation for $p_u(a, z | \bar{z})$ into the above to get

$$p_u(a, z, \bar{z}) = p_u(a, z | \bar{z}) \frac{\left[\frac{1 - e^{-\theta_*(\bar{z} - z_d)}}{\kappa} \right] g(\bar{z})}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z} - z_d)}}{\kappa} \right] g(\bar{z}) d\bar{z} \right]}$$

Now integrate with respect to a and over \bar{z} in the region (z_d, ∞) to get

$$p_u(z) = \frac{\int_{z_d}^{\infty} p_u(z | \bar{z}) \left[\frac{1 - e^{-\theta_*(\bar{z} - z_d)}}{\kappa} \right] g(\bar{z}) d\bar{z}}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z} - z_d)}}{\kappa} \right] g(\bar{z}) d\bar{z} \right]}$$

A.3 Hysteresis Derivations

A.3.1 System of Equations

The system of equations that implicitly defines the threshold values z_l and z_h and the constants a_1 and b_2 is

$$\begin{aligned} b_2 e^{\beta_2 z_h} - \frac{\tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1} P)^{\sigma-1}}{\frac{1}{2} \xi^2 (\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z_h} - \frac{f_x}{(r+\kappa)} &= a_1 e^{\beta_1 z_h} + f_h \\ \beta_2 b_2 e^{\beta_2 z_h} - \frac{\tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1} P)^{\sigma-1} (\sigma-1)}{\frac{1}{2} \xi^2 (\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z_h} &= \beta_1 a_1 e^{\beta_1 z_h} \\ b_2 e^{\beta_2 z_l} - \frac{\tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1} P)^{\sigma-1}}{\frac{1}{2} \xi^2 (\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z_l} - \frac{f_x}{(r+\kappa)} &= a_1 e^{\beta_1 z_l} \\ \beta_2 b_2 e^{\beta_2 z_l} - \frac{\tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1} P)^{\sigma-1} (\sigma-1)}{\frac{1}{2} \xi^2 (\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z_l} &= \beta_1 a_1 e^{\beta_1 z_l} \end{aligned}$$

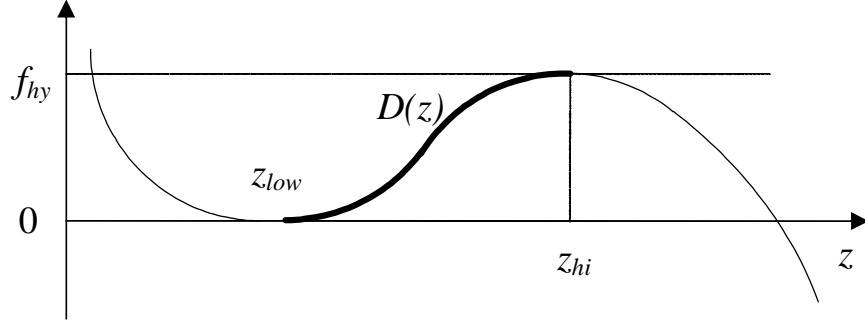


Figure 9: Firm's Incremental Value of Becoming an Exporter over (z_{low}, z_{hi})

A.3.2 Properties of z_l and z_h

Exporting Cutoff (z_h) Define the function,

$$\begin{aligned} D(z) &\equiv V_h(z) - V_l(z) \\ &= b_2 e^{\beta_2 z} - a_1 e^{\beta_1 z} - \frac{\tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1} P)^{\sigma-1}}{\frac{1}{2} \xi^2 (\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z} - \frac{f_x}{(r+\kappa)} \end{aligned}$$

that can be interpreted as the firm's incremental value of becoming an exporter, over the range (z_l, z_h) .¹⁴ When z is small, the dominant term in $D(z)$ is the one with the negative root β_2 . It is decreasing and convex in z . When z is large the dominant term is the one with the positive root β_1 .¹⁵ This term is negative, decreasing and concave. For intermediate values, the third term contributes to the increasing portion of $D(z)$ (see Figure (9)).

Consider the upper threshold z_h . Subtracting the differential equation for V_l from the one for V_h , we have

$$(r+\kappa)D(z) = \pi_{ex}(z) + \alpha D'(z) + \frac{1}{2} \xi^2 D''(z)$$

¹⁴Here we follow a similar case illustrated in Dixit and Pindyck (1994).

¹⁵This is guaranteed by Assumption 1, which by stating that $\rho + \delta > \alpha(\sigma-1) + \xi^2(\sigma-1)^2/2$, actually impose that $(\sigma-1) < \beta_1$. To prove it just solve the inequality for $(\sigma-1)$ and find that the solutions gives $|\sigma-1| < \beta_1$. Recall also that demand is elastic so that $\sigma > 1$.

Evaluating at z_h and using the boundary conditions that must hold at z_h , we get

$$-(r + \kappa)f_h + \pi_{ex}(z_h) = -\frac{1}{2}\xi^2 D''(z_h) > 0$$

or $\pi_{ex}(z_h) > (r + \kappa)f_h > 0$ which means that z_h is larger than the productivity level at which the firm decides to become an exporter when there is no uncertainty and z is constant over time.

Width of the Band of Inaction Define, over the range (z_l, z_h) ,

$$\begin{aligned} D(z) &= V_h(z) - V_l(z) \\ &= b_2 e^{\beta_2 z} - a_1 e^{\beta_1 z} - \frac{\tau^{1-\sigma} \frac{R}{\sigma} (\bar{m}^{-1} P)^{\sigma-1}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\kappa)} e^{(\sigma-1)z} - \frac{f_x}{(r+\kappa)} \end{aligned}$$

Write the value-matching and smooth-pasting conditions in terms of $D(z)$,

$$\begin{aligned} D(z_h, a_1, b_2) &= f_h \\ D_z(z_h, a_1, b_2) &= 0 \\ D(z_l, a_1, b_2) &= 0 \\ D_z(z_l, a_1, b_2) &= 0 \end{aligned}$$

We are going to find out what is the effect of a small change in f_h on the cutoff thresholds z_l and z_h . First, totally differentiate the value-matching conditions

$$D_z(z_h, a_1, b_2)dz_h + D_{a_1}(z_h, a_1, b_2)da_1 + D_{b_2}(z_h, a_1, b_2)db_2 = df_h$$

$$D_z(z_l, a_1, b_2)dz_l + D_{a_1}(z_l, a_1, b_2)da_1 + D_{b_2}(z_l, a_1, b_2)db_2 = 0$$

which, using the smooth-pasting condition, simplify to

$$D_{a_1}(z_h, a_1, b_2)da_1 + D_{b_2}(z_h, a_1, b_2)db_2 = df_h$$

$$D_{a_1}(z_l, a_1, b_2)da_1 + D_{b_2}(z_l, a_1, b_2)db_2 = 0$$

and

$$-e^{\beta_1 z_h} da_1 + e^{\beta_2 z_h} db_2 = df_h$$

$$-e^{\beta_1 z_l} da_1 + e^{\beta_2 z_l} db_2 = 0$$

Solving the system we find that $db_2 = e^{(\beta_1 - \beta_2)z_l} da_1$ and that $da_1 = \chi df_h$

where $\chi = (e^{\beta_2(z_h - z_l)} e^{\beta_1 z_l} - e^{\beta_1 z_h})^{-1} < 0$ since $e^{\beta_2(z_h - z_l)} < 1$, $\beta_1 > 0$ and $z_h > z_l$.

Now differentiate the first smooth-pasting condition at z_h

$$D_{zz}(z_h, a_1, b_2)dz_h + D_{za_1}(z_h, a_1, b_2)da_1 + D_{zb_2}(z_h, a_1, b_2)db_2 = 0$$

that, after using the expressions for da_1 and db_2 yields

$$\begin{aligned} D_{zz}(z_h, a_1, b_2)dz_h &= \beta_1 e^{\beta_1 z_h} da_1 - \beta_2 e^{\beta_2 z_h} db_2 \\ &= \frac{[\beta_1 e^{\beta_1 z_h} - \beta_2 e^{\beta_2 z_h + (\beta_1 - \beta_2)z_l}]}{(e^{\beta_2(z_h - z_l)} e^{\beta_1 z_l} - e^{\beta_1 z_h})} df_h \end{aligned}$$

Recall that $D(z)$ is concave at z_h so that $D_{zz}(z_h, a_1, b_2) < 0$. Note that the term in front of df_h is negative as well so that $dz_h/df_h > 0$. When the cost of entering the export market is higher, the export cutoff is also higher.

Now differentiate the second smooth-pasting condition at z_l

$$D_{zz}(z_l, a_1, b_2)dz_l + D_{za_1}(z_l, a_1, b_2)da_1 + D_{zb_2}(z_l, a_1, b_2)db_2 = 0$$

which yields

$$\begin{aligned} D_{zz}(z_l, a_1, b_2)dz_l &= \beta_1 e^{\beta_1 z_l} da_1 - \beta_2 e^{\beta_2 z_l} db_2 \\ &= \frac{[\beta_1 e^{\beta_1 z_l} - \beta_2 e^{\beta_2 z_l + (\beta_1 - \beta_2)z_l}]}{(e^{\beta_2(z_h - z_l)} e^{\beta_1 z_l} - e^{\beta_1 z_h})} df_h \end{aligned}$$

Recall that $D(z)$ is convex at z_l so that $D_{zz}(z_l, a_1, b_2) > 0$. Note that the term in front of df_h is still negative so that $dz_l/df_h < 0$. When the cost of entering the export market is higher, the export-abandon cutoff is lower. This proves that the width of the band of inaction is an increasing function of f_h .

A.3.3 Density of Exporters

Consider the solution for $v(a, z|\bar{z})$ that we derived previously:

$$\begin{aligned} v(a, z|\bar{z}) &= u(a, z - (z_l - z_d)|\bar{z} - (z_l - z_d))1_{\{\bar{z} > z_h\}} \\ &\quad - \frac{1}{2} \xi^2 \frac{K'_v(z_h^-, \bar{z}) - K'_v(z_h^-, \bar{z})1_{\{\bar{z} > z_h\}}}{1 - \frac{\xi^2}{2} K'_v(z_h^-, z_h)} u(a, z - (z_l - z_d)|z_h - (z_l - z_d)) \end{aligned}$$

The boundary condition is

$$\lim_{a \downarrow 0} \int_{z_h}^z v(a, t, \bar{z}) dt = \begin{cases} g_v(\bar{z}) \frac{M_a}{M_x} & z \geq \bar{z} \\ 0 & z < \bar{z} \end{cases}$$

where M_x is the number of exporters. The stationary density for exporters is

$$v(a, z, \bar{z}) = e^{-\kappa a} v(a, z | \bar{z}) g_u(\bar{z}) \frac{M_a}{M_x}$$

Note that some exporters may have started as nonexporters and for this reason $v(a, z, \bar{z})$ includes $g_u(\bar{z})$ and not $g_v(\bar{z})$.

Now let's derive the pdf $p_v(z | \bar{z})$ and $p_v(a, z | \bar{z})$ from $\check{v}(a, z, | \bar{z}) = e^{-\kappa a} v(z, a | \bar{z})$.

$$\begin{aligned} \check{v}(z | \bar{z}) &= \int_0^\infty \check{v}(a, z, | \bar{z}) da \\ &= m_{1\bar{z}} - e^{-\frac{2\alpha}{\xi^2}(\bar{z}-z_l)} m_{2\bar{z}} - \Upsilon(m_{1z_h} - e^{-\frac{2\alpha}{\xi^2}(z_h-z_l)} m_{2z_h}) \end{aligned}$$

where $\Upsilon = \frac{1}{2} \xi^2 \frac{K'(z_h^-, \bar{z}) - K'_v(z_h^-, \bar{z})}{1 - \frac{\xi^2}{2} K'_v(z_h^-, z_h)}$, $m_{1t} = \frac{\min\{e^{\theta_*(z-t)}, e^{-\theta(z-t)}\}}{\sqrt{\alpha^2 + 2\xi^2\kappa}}$ and $m_{2t} = \frac{\exp(-\theta[z+t-2z_l])}{\sqrt{\alpha^2 + 2\xi^2\kappa}}$.

Combining these results,

$$\begin{aligned} \check{v}(z | \bar{z}) &= -\frac{1}{\alpha} \frac{\theta - \theta_*}{\theta + \theta_*} \frac{[\min\{e^{(\theta+\theta_*)(z-z_l)}, e^{(\theta+\theta_*)(\bar{z}-z_l)}\} - 1]}{e^{\theta_*(\bar{z}-z_l)} e^{\theta(z-z_l)}} \\ &\quad + \frac{\Upsilon}{\alpha} \frac{\theta - \theta_*}{\theta + \theta_*} \frac{[\min\{e^{(\theta+\theta_*)(z-z_l)}, e^{(\theta+\theta_*)(z_h-z_l)}\} - 1]}{e^{\theta_*(z_h-z_l)} e^{\theta(z-z_l)}} \end{aligned}$$

Now we can find the normalizing constant,

$$\int_{z_l}^\infty \check{v}(z | \bar{z}) dz = -\frac{1}{\alpha} \frac{\theta - \theta_*}{\theta \theta_*} \left[1 - e^{-\theta_*(\bar{z}-z_l)} - \Upsilon(1 - e^{-\theta_*(z_h-z_l)}) \right]$$

Manipulating the constants we get

$$-\frac{1}{\alpha} \frac{\theta - \theta_*}{\theta \theta_*} = \frac{1}{\kappa}$$

so that,

$$\int_{z_l}^\infty \check{v}(z | \bar{z}) dz = \frac{1}{\kappa} \left[1 - e^{-\theta_*(\bar{z}-z_l)} - \Upsilon(1 - e^{-\theta_*(z_h-z_l)}) \right]$$

The probability density $p_v(z|\bar{z})$ is thus:

$$p_v(z|\bar{z}) = \frac{\check{v}(z|\bar{z})}{\int_{z_l}^{\infty} \check{v}(z|\bar{z})dz} = \frac{\theta\theta_*}{\theta + \theta_*} \left\{ \frac{\frac{\min\{e^{(\theta+\theta_*)(z-z_l)}, e^{(\theta+\theta_*)(\bar{z}-z_l)}\}-1}{e^{\theta(z-z_l)}(e^{\theta_*(\bar{z}-z_l)}-1)-\Upsilon e^{\theta_*(\bar{z}-z_h)}e^{\theta(z-z_l)}(e^{\theta_*(z_h-z_l)}-1)}-1}{-\Upsilon \frac{\min\{e^{(\theta+\theta_*)(z-z_l)}, e^{(\theta+\theta_*)(z_h-z_l)}\}-1}{[e^{\theta_*(z_h-\bar{z})}e^{\theta(z-z_l)}(e^{\theta_*(\bar{z}-z_l)}-1)-\Upsilon e^{\theta(z-z_l)}(e^{\theta_*(z_h-z_l)}-1)]}} \right\}$$

and the probability density $p_v(a, z|\bar{z})$ is,

$$p_v(a, z|\bar{z}) = \frac{e^{-\kappa a}\psi_v(z, a|\bar{z})}{\frac{1}{\kappa} [1 - e^{-\theta_*(\bar{z}-z_l)} - \Upsilon(1 - e^{-\theta_*(z_h-z_l)})]}$$

Now we can derive $\frac{M_a}{M_x}$. Consider the stationary distribution

$$v(a, z, \bar{z}) = e^{-\kappa a}v(a, z|\bar{z})g_u(\bar{z})\frac{M_a}{M_x}$$

The latter is a probability density for a particular value of M_a/M_x . This is used to

determine the amount of entry that must take place relative to the number of existing firms. We will use the fact that $v(a, z|\bar{z})$ is a probability density function.

$$1 = \int_{z_h}^{\infty} \int_{z_l}^{\infty} \int_0^{\infty} v(a, z, \bar{z})dadz d\bar{z} = \int_{z_h}^{\infty} \int_{z_l}^{\infty} \int_0^{\infty} e^{-\kappa a}\psi_v(a, z|\bar{z})g_u(\bar{z})\frac{M_a}{M_x}dadz d\bar{z}$$

Replace the definition of $v(a, z|\bar{z})$,

$$1 = \int_{z_h}^{\infty} \int_{z_l}^{\infty} \int_0^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_l)} - \Upsilon(1 - e^{-\theta_*(z_h-z_l)})}{\kappa} \right] v(a, z|\bar{z})g_u(\bar{z})\frac{M_a}{M_x}dadz d\bar{z}$$

so that the equilibrium rate of entry is

$$\frac{M_a}{M_x} = \left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_l)} - \Upsilon(1 - e^{-\theta_*(z_h-z_l)})}{\kappa} \right] g_u(\bar{z})d\bar{z} \right]^{-1}$$

Now we can finally derive $v(z)$.

Use the attempted entry rate equation to get the probability density

$$v(a, z, \bar{z}) = \frac{e^{-\kappa a}\psi_v(a, z|\bar{z})g_u(\bar{z})}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_l)} - \Upsilon(1 - e^{-\theta_*(z_h-z_l)})}{\kappa} \right] g_u(\bar{z})d\bar{z} \right]}$$

and replace the equation for $v(a, z|\bar{z})$ into the above to get

$$v(a, z, \bar{z}) = v(a, z|\bar{z}) \frac{\left[\frac{1 - e^{-\theta_*(\bar{z} - z_l)} - \Upsilon(1 - e^{-\theta_*(z_h - z_l)})}{\kappa} \right] g_u(\bar{z})}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z} - z_l)} - \Upsilon(1 - e^{-\theta_*(z_h - z_l)})}{\kappa} \right] g_u(\bar{z}) d\bar{z} \right]}$$

Now integrate with respect to a and over \bar{z} in the region (z_h, ∞) to get

$$v(z) = \frac{\int_{z_h}^{\infty} v(z|\bar{z}) \left[\frac{1 - e^{-\theta_*(\bar{z} - z_l)} - \Upsilon(1 - e^{-\theta_*(z_h - z_l)})}{\kappa} \right] g_u(\bar{z}) d\bar{z}}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z} - z_l)} - \Upsilon(1 - e^{-\theta_*(z_h - z_l)})}{\kappa} \right] g_u(\bar{z}) d\bar{z} \right]}$$