

# A Differential Game with Investment in Transport and Communication R&D

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## Abstract

We analyse R&D activity in transport and communication technology, in a differential duopoly game where firms compete, alternatively, in prices or quantities. Transport and communication costs are of the iceberg type, i.e., a portion of the product is lost during the transportation phase. Firms invest to increase the net amount of the product that reaches consumers. We derive subgame perfect equilibria, and show that price competition yields the socially optimal investment, while Cournot competition involves excess investment.

**JEL classification:** D43, L13, O31.

**Keywords:** R&D, differential games, transport and communication costs.

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# 1 Introduction

Transport and communication technologies (TC) have always represented a relevant feature in the literature on trade (see Helpman and Krugman, 1985, *inter alia*). However, most of the works in this field tend to consider the transportation plan and its related technology as fixed characteristics.

Even if firms devote substantial amounts on research and development (R&D), only process innovation and product innovation have been widely explored. The activity of product innovation consists in the development of technologies for producing new products or for increasing the quality of the existing ones. On the other hand, process innovation aims at decreasing the costs of producing existing products. Literature has considered the different degree of efficiency of process innovating R&D between the Cournot and the Bertrand setting. An established result states that there is an excess of process-innovating R&D under Cournot competition, while the opposite holds under Bertrand competition (Brander and Spencer, 1983; Dixon, 1985).

A more comprehensive analysis requires then the study of R&D activities that allow firms to reach markets in a more efficient way and be more competitive in serving their customers. Launhardt (1885), whose contribution has been recently acknowledged, proposed a simple spatial duopoly model with both horizontal and vertical product differentiation. Furthermore, he paid attention to the influence of differences in transportation costs. He thus recognized the possibility of different form of heterogeneity among firms, associated either to location or to transportation technology. Recently, Thisse and Dos Santos Ferreira (1996) expanded Launhardt model by allowing firms to choose their transportation cost technologies.

So far, however, the topic of strategic investment to reduce the burden of TC costs has been rather neglected. In this paper we will then consider investments in R&D that concern mainly transport and communication needed to let the product reach the final buyer. The related R&D may be figured out as an expenditure that is going to improve the technology of the last stage of the production process (investment in the Internet, in advanced logistics, or in faster transport technology). We define this sort of activity transport and communication R&D (TCRD). Transport and communication costs are assumed to be of the ‘iceberg’ form invented by Samuelson (1954) and widely used in trade theory (Helpman and Krugman, 1985; Krugman, 1990). When a quantity  $q_i$  of product  $i$  is produced, yet only a portion  $q_i/s_i$ ,  $s_i > 1$  of the product reaches the consumer. By investing in communication and transport specific R&D, a firm may increase such a portion thus enlarging its market share.

Lambertini, Mantovani and Rossini (2003) analyse R&D activity in transport and communication technology (TCRD), in a static Cournot duopoly. They find that a variety of equilibria may appear as a result of the different levels of TCRD efficiency. If the analysis is extended to a continuous choice space, equilibria exist only if production costs are low *vis à vis* market size and transport costs. Finally, Lambertini and Rossini (2001) investigate the role of TCRD in a Cournot duopoly with trade.

We propose a differential game where firms invest to increase the percentage of the product that arrives at destination. Although, as to our knowledge, most of the applications of differential games can be found in different fields of industrial organization (see Dockner *et al.*, 2000; Cellini and Lambertini, 2003), the problem of TCRD investments has never been considered in such a framework.

We present a dynamic model of duopoly with differentiated products, where firms

compete either in prices or quantities in a unique market and invest in TCRD. Alternatively, as usual in the international trade literature (see Spencer and Brander, 1983, *inter alia*), one can think of two firms located in different countries that export to a third market. We consider both open-loop and closed loop Nash equilibria, with a closer attention to closed loop ones. Finally, we compare the solutions appearing in the previous cases with the social optimum and proceed with a welfare appraisal. This allows to draw some conclusions on the efficiency of Bertrand and Cournot oligopolies. In particular, in line with the existing literature, we prove that Bertrand competition yields the socially optimal amount of TCRD effort, while Cournot competition involves excess investment.

The paper is organized as follows. The basic model is laid out in Section 1. Section 2 considers Bertrand competition while Section 3 deals with the Cournot setting. Section 4 deals with the social optimum and the welfare appraisal. Section 5 gives the conclusion.

## 2 The model

We employ a quadratic utility function for a representative consumer as in Bowley (1924), Dixit (1979) and Singh and Vives (1984):

$$U(t) = A\hat{q}_1(t) + A\hat{q}_2(t) - \frac{1}{2} [\hat{q}_1(t)^2 + 2\gamma\hat{q}_1(t)\hat{q}_2(t) + \hat{q}_2(t)^2] \quad (1)$$

whose maximization under the budget constraint  $Y(t) \geq \sum_i p_i(t)\hat{q}_i(t)$ ,  $i = \{1, 2\}$  where  $Y(t)$  is nominal income, yields demand functions:

$$p_i(t) = A - \hat{q}_i(t) - \gamma\hat{q}_j(t) \quad \forall i \neq j, \quad i, j = \{1, 2\} \quad (2)$$

where  $\gamma \in [0, 1]$  stands for the symmetric degree of substitutability between the two goods and  $A$  is the market-size, both supposed to be constant over time;  $\hat{q}_i(t) \equiv \frac{q_i(t)}{s_i(t)}$  represents the share of firm  $i$ 's good that is available for consumption at price  $p_i(t)$ , i.e.,  $1 - 1/s_i(t)$ , with  $s_i(t) > 1 \forall t \in [0, \infty)$ , indicates the fraction of firm  $i$ 's good that is lost during the transportation phase (see Samuelson, 1954).

The direct demand functions can be obtained by solving (2) for  $q_i(t)$ :

$$q_i(t) = \frac{s_i(t) [A(\gamma - 1) + p_i(t) - \gamma p_j(t)]}{\gamma^2 - 1} \quad (3)$$

On the supply side, we make the assumption that production entails constant marginal costs, normalized to zero for the sake of simplicity. Instantaneous profits are then given by:

$$\pi_i(t) = \frac{q_i(t)}{s_i(t)} p_i(t) - \beta [k_i(t)]^2 \quad (4)$$

where  $k_i(t)$  represents the amount of effort made by firm  $i$  at time  $t$  in order to reduce  $s_i(t)$  and parameter  $\beta$  is an inverse measure of TCRD productivity. By reducing  $s_i(t)$  through capital accumulation over time, firm  $i$  increases the fraction of  $q_i(t)$  that reaches the final market. We assume that  $s_i(t)$  evolves over time according to the following kinematic equation:

$$\frac{\partial s_i(t)}{\partial t} = [\alpha k_i(t) - \delta s_i(t)] [1 - s_i(t)] \quad (5)$$

where  $\delta$  denotes the depreciation rate, which is common to both firms and constant over time;  $\alpha$  is a time-invariant parameter positively affecting the accumulation process. It is worth noting that (5) accounts for the fact that  $s_i(t)$  has to be always greater than unity for our model to be meaningful: when  $s_i(t) = 1$  what is produced corresponds to what is offered, and capital accumulation stops, otherwise an increase in the capital stock yields a decrease in  $s_i(t)$  as long as  $k_i(t) > \frac{\delta}{\alpha} s_i(t)$ .

We assume that the two firms behave alternatively as either price or quantity setters. Each firm  $i$  aims at maximizing the discounted profit flow:

$$\Pi_i(t) = \int_0^\infty \pi_i(t) e^{-\rho t} dt \quad (6)$$

w.r.t. controls  $k_i(t)$  and the market variable, either  $p_i(t)$  or  $q_i(t)$ , under the constraint given by the state dynamics (5). The discount rate  $\rho > 0$  is assumed to be constant and common to both firms. The corresponding current value Hamiltonian function is:

$$\mathcal{H}_i(t) = e^{-\rho t} [\pi_i(t) + \lambda_{ii}(t)\dot{s}_i(t) + \lambda_{ij}(t)\dot{s}_j(t)] \quad (7)$$

where  $\lambda_{ii}(t) = \mu_{ii}(t)e^{\rho t}$  and  $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$ ,  $\mu_{ii}(t)$  being the co-state variable associated to  $s_i(t)$ .

For future reference, we also define consumer surplus  $CS(t) \equiv U(t) - \sum_i p_i(t) \hat{q}_i(t)$ . Under the symmetry assumption  $\hat{q}_i(t) = \hat{q}_j(t) = \hat{q}(t) = q(t)/s$ , consumer surplus writes as:

$$CS(t) = \hat{q}(t)^2 [1 + \gamma] = \left[ \frac{q(t)}{s(t)} \right]^2 [1 + \gamma]. \quad (8)$$

Disregarding the issue of surplus distribution among agents, we can define social welfare as  $SW(t) \equiv 2\pi(t) + CS(t)$ , with  $\pi(t) = q(t)\{(As(t) - q(t)[1 + \gamma])/[s(t)]^2\}$ :

$$SW(t) = 2A\hat{q}(t) - \hat{q}(t)^2 [1 + \gamma] = 2A\frac{q(t)}{s(t)} - \left[ \frac{q(t)}{s(t)} \right]^2 [1 + \gamma]. \quad (9)$$

Note that (9) is decreasing in  $s$  as long as  $q < As/(1 + \gamma)$ .

### 3 Bertrand competition

Let us move from the case of firms competing in prices. By substituting (3) in (6) we get the relevant objective function for firm  $i$ :

$$\Pi_i(t) = \int_0^\infty e^{-\rho t} \left\{ \left[ \frac{A(1 - \gamma) - p_i(t) + \gamma p_j(t)}{1 - \gamma^2} \right] p_i(t) - \beta [k_i(t)]^2 \right\} dt \quad (10)$$

Using the Hamiltonian function (7), first order conditions on controls are (we omit the indication of time for brevity):

$$\frac{\partial \mathcal{H}_i}{\partial p_i} = 0 \Rightarrow p_i = \frac{A(1-\gamma) + \gamma p_j}{2} \quad (11)$$

$$\frac{\partial \mathcal{H}_i}{\partial k_i} = 0 \Rightarrow \lambda_{ii} = \frac{2\beta k_i}{\alpha(1-s_i)} \quad (12)$$

Optimal prices can be easily derived from (11) by imposing symmetry ( $p_i = p_j$ ):

$$p^B = \frac{A(1-\gamma)}{2-\gamma} \quad (13)$$

which can be plugged into (3) to get optimal quantities:

$$q^B = \frac{A}{(\gamma+1)(2-\gamma)} \quad (14)$$

Note, first, that (12) does not contain  $\lambda_{ij}$  since the present game features separated dynamics. Therefore, we set  $\lambda_{ij} = 0$  for all  $t \in [0, \infty)$ . Secondly, (11) does not contain  $s_j$ , therefore the open-loop solution and the closed-loop memoryless solution coincide. Second order conditions are always met throughout the paper. They are omitted for brevity.

### 3.1 Degenerate Markov Perfect Nash Equilibrium

According to the closed-loop memoryless solution concept, we specify the firm  $i$ 's co-state equation as follows:

$$-\frac{\partial \mathcal{H}_i}{\partial s_i} = \lambda_{ii}(\alpha k_i + \delta - 2\delta s_i) = \dot{\lambda}_{ii} - \rho \lambda_{ii} \quad (15)$$

along with the transversality and initial conditions:

$$\lim_{t \rightarrow \infty} \mu_i s_i = 0, \quad s_i(0) > 1. \quad (16)$$

Now, by using (12), (11) and the co-state equation (15), we write:

$$\dot{k}_i = k_i (\rho + \delta - \delta s_i) \quad (17)$$

The steady state equilibrium requires  $\{\dot{k}_i = 0, \dot{s}_i = 0\}$ , yielding:

$$k_i^{BSS} = \frac{\rho + \delta}{\alpha} > 0; \quad s_i^{BSS} = \frac{\rho + \delta}{\delta} > 1 \quad (18)$$

We are interested in the dynamics of the system formed by (17) and (5) in the  $\{k_i, s_i\}$  space, which can be represented in Figure 1:

*INSERT FIGURE 1 HERE*

The above phase diagram suggests that the equilibrium is a saddle, and that it can be approached along the north-east arm of the path. It is worth noting that if  $s_i(0) \gg \frac{\rho + \delta}{\delta}$ , then the system never converges to the equilibrium, the reason being that the effort required to increase the share of the good arriving at destination is too costly. We are now in a position to write:

**Proposition 1** *The steady state defined by  $\{k_i^{BSS}, s_i^{BSS}\}$  is a saddle point.*

**Proof.** See the Appendix. ■

As to the comparative statics of the steady state w.r.t. all involved parameters, we have the following properties:  $\frac{\partial s_i^{BSS}}{\partial \rho} > 0$  and  $\frac{\partial s_i^{BSS}}{\partial \delta} < 0$ . First, when the rate of time preference  $\rho$  increases, firms invest more at the steady state level but a lower fraction arrives at destination. Firms are impatient to deliver a higher fraction and spend substantial amounts on TCRD without waiting for the beneficial effect coming from the accumulation dynamics. As to  $\delta$ , at the steady state a higher depreciation rate increases the level of capital and consequently the fraction of output that reaches

the market. To verify the intuition behind this, observe that:

$$\frac{\partial s_i}{\partial \delta} = \frac{\partial s_i}{\partial k_i} \cdot \frac{\partial k_i}{\partial \delta}$$

where:

$$\frac{\partial s_i}{\partial k_i} < 0 \text{ and } \frac{\partial k_i}{\partial \delta} > 0.$$

## 4 Cournot Competition

The relevant objective function for firm  $i$  in case of quantity competition is:

$$\Pi_i = \int_0^\infty e^{-\rho t} \left\{ \frac{q_i}{s_i} \left[ A - \frac{q_i}{s_i} - \gamma \frac{q_j}{s_j} \right] - \beta [k_i]^2 \right\} dt \quad (19)$$

Using the Hamiltonian function (7), first order conditions on controls are:

$$\frac{\partial \mathcal{H}_i}{\partial q_i} = 0 \Rightarrow q_i = \frac{s_i (A s_j - \gamma q_j)}{2 s_j} \quad (20)$$

$$\frac{\partial \mathcal{H}_i}{\partial k_i} = 0 \Rightarrow \lambda_{ii} = \frac{2 \beta k_i}{\alpha (1 - s_i)} \quad (21)$$

One can find optimal quantities by imposing symmetry ( $q_i = q_j$ ;  $s_i = s_j$ ) and by rearranging (20):

$$q^C = \frac{A s}{2 + \gamma} \quad (22)$$

Optimal prices are then given by:

$$p^C = \frac{A}{2 + \gamma} \quad (23)$$

Note that, first, (21) is equivalent to (12), so it does not contain  $\lambda_{ij}$  (we set  $\lambda_{ij} = 0$  for all  $t \in [0, \infty)$ ). However, (20) contains  $s_j$ , i.e., the state variable of the rival, meaning that the open-loop solution and the closed-loop memoryless solution does not coincide anymore. As a consequence, we deal with the two solution concepts.

## 4.1 Open-Loop Nash Equilibrium

Under the open-loop solution concept, we can specify the firm  $i$ 's co-state equation as follows:

$$-\frac{\partial \mathcal{H}_i}{\partial s_i} = \frac{2q_i^2}{s_i^3} + \frac{q_i(\gamma q_j - A s_j)}{s_i^2 s_j} - \lambda_{ii}(\alpha k_i + \delta - 2\delta s_i) = \dot{\lambda}_{ii} - \rho \lambda_{ii} \quad (24)$$

along with the transversality and initial conditions:

$$\lim_{t \rightarrow \infty} \mu_i s_i = 0, \quad s_i(0) > 1. \quad (25)$$

Now, by using (21), (20) and the co-state equation (24), we write:

$$\dot{k}_i = k_i(\rho + \delta - \delta s_i) \quad (26)$$

The steady state equilibrium requires  $\{\dot{k}_i = 0, \dot{s}_i = 0\}$ , yielding:

$$k_i^{OL} = \frac{\rho + \delta}{\alpha}; \quad s_i^{OL} = \frac{\rho + \delta}{\delta} \quad (27)$$

It is evident that (27) are the same as (18), albeit it may be quickly checked that equilibrium profits are different. Nonetheless, such a comparison has a limited interest in that it involves open loop solutions, which are only weakly time consistent.

## 4.2 Closed-Loop Nash Equilibrium

In order to perform a meaningful comparison between market regimes, we need to solve the game in closed-loop, taking into account the feedback between player  $i$ 's strategy and player  $j$ 's state variable. This will lead to an equilibrium characterized by subgame perfection.

We specify the firm  $i$ 's co-state equation:

$$-\frac{\partial \mathcal{H}_i}{\partial s_i} - \frac{\partial \mathcal{H}_i}{\partial q_j} \frac{\partial q_j^*}{\partial s_i} \equiv \Phi = \dot{\lambda}_{ii} - \rho \lambda_{ii} \quad (28)$$

with

$$\Phi = \frac{2q_i^2}{s_i^3} + \frac{q_i(\gamma q_j - A s_j)}{s_i^2 s_j} - \lambda_{ii}(\alpha k_i + \delta - 2\delta s_i) - \left(-\frac{\gamma q_i}{s_i s_j}\right) \left(\frac{\gamma q_i s_j}{2s_i^2}\right)$$

along with the transversality and initial conditions:

$$\lim_{t \rightarrow \infty} \mu_i s_i = 0, \quad s_i(0) > 1. \quad (29)$$

Now, by using (21), (20) and the co-state equation (28), we write:

$$\dot{k}_i = \frac{-A^2 \gamma^2 (s_i - 1) \alpha + 4\beta (2 + \gamma)^2 k_i s_i (\rho - \delta s_i + \delta)}{4\beta (2 + \gamma)^2 s_i} \quad (30)$$

The steady state conditions  $\{\dot{k}_i = 0, \dot{s}_i = 0\}$  yield the following real solutions:

$$k_i^{CL} = \frac{1}{6\alpha} \left[ 2(\delta + \rho) + \frac{3A^2 \gamma^2 \alpha^2 - 4\beta (2 + \gamma)^2 (\delta + \rho)^2}{\Psi} - \frac{\Psi}{\beta (2 + \gamma)^2} \right] \quad (31)$$

$$s_i^{CL} = \frac{\alpha}{\delta} k_i^{CL} \quad (32)$$

where the exact value of  $\Psi$  is provided in the Appendix.

**Proposition 2** *The steady state defined by  $\{k_i^{CL}, s_i^{CL}\}$  is a saddle point.*

**Proof.** See the Appendix.

However, (31) is not manageable in comparing market regimes. Therefore, we proceed as follows. We impose  $\dot{k}_i = 0$  to determine an equilibrium relation between  $k_i^{CL}$  and  $s_i^{CL}$ :

$$k_i^{CL}(s_i^{CL}) = \frac{A^2 \gamma^2 (s_i^{CL} - 1) \alpha}{4\beta (2 + \gamma)^2 s_i^{CL} (\rho - \delta s_i^{CL} + \delta)} \quad (33)$$

We are now in a position to compare the optimal open-loop and closed-loop level of R&D investments under Cournot competition. This is equivalent to compare the closed-loop solution arising in Cournot *vis à vis* Bertrand's, given that the open-loop

solution with quantities as a control variable corresponds to the closed-loop solution with prices as a control variable.

From a direct comparison between (27) and (33), we have:

**Proposition 3** *The optimal effort in TCRD is higher under Cournot competition than under Bertrand competition.*

**Proof.**  $k_i^{CL}(s_i) = \frac{A^2 \gamma^2 (s_i - 1) \alpha}{4b(2 + \gamma)^2 s_i (\rho - \delta s_i + \delta)} > 0$  to be acceptable. Since the numerator is always positive by definition, it has to be true that  $\rho - \delta s_i + \delta > 0 \Rightarrow s_i < \frac{\rho + \delta}{\delta} \equiv s_i^{OL}$ . This amounts to saying that  $k_i^{CL} > k_i^{OL}$ . ■

This result is in line with the kind of R&D activity at stake, which aims at increasing the percentage of the produced good that reaches the market. Moreover, we confirm the conventional wisdom that firms invest more when using closed-loop decision rules than open-loop ones.

## 5 Social Optimum and Welfare Appraisal

The aim of this section is threefold: (i) to characterize the first best solution; (ii) to compare the private optima with the social optimum; (iii) to compare the welfare generated by Bertrand competition with that generated by Cournot competition. The objective function of an hypothetical benevolent planner is:

$$SW(t) = \int_0^\infty e^{-\rho t} \left\{ 2A \frac{q(t)}{s(t)} - \left[ \frac{q(t)}{s(t)} \right]^2 (1 + \gamma) - 2\beta [k(t)]^2 \right\} dt \quad (34)$$

to be maximized w.r.t.  $q(t)$  and  $k(t)$  under the dynamic constraint:

$$\frac{\partial s(t)}{\partial t} = [\alpha k(t) - \delta s(t)] [1 - s(t)] \quad (35)$$

The current value Hamiltonian function writes:

$$\mathcal{H}^{SP}(t) = e^{-\rho t} [SW(t) + \lambda(t)\dot{s}(t)] \quad (36)$$

First-order conditions on controls are (we omit the indication of time for brevity):

$$\frac{\partial \mathcal{H}^{SP}}{\partial q} = 0 \Rightarrow q = \frac{As}{1 + \gamma} \quad (37)$$

$$\frac{\partial \mathcal{H}^{SP}}{\partial k} = 0 \Rightarrow \lambda = \frac{4\beta k}{\alpha(1 - s)} \quad (38)$$

$$-\frac{\partial \mathcal{H}^{SP}}{\partial s} = \frac{q[2As - 2q(1 + \gamma)] + \lambda s^3[\delta(1 - 2s) + \alpha k]}{s^3} = -\rho\lambda + \dot{\lambda} \quad (39)$$

By differentiating (38) w.r.t. time we obtain:

$$\dot{\lambda} = \frac{4\beta(\dot{k}(1 - s) + k\dot{s})}{\alpha(1 - s)^2} \quad (40)$$

By plugging (40), (38) and (37) into (39) and by using (35) we have:

$$\dot{k} = k_i(\rho + \delta - \delta s_i) \quad (41)$$

The steady state equilibrium requires  $\{\dot{k} = 0, \dot{s} = 0\}$ , yielding:

$$k^{SP} = \frac{\rho + \delta}{\alpha}; \quad s^{SP} = \frac{\rho + \delta}{\delta} \quad (42)$$

On the basis of the steady state solutions previously obtained, without further proof, we can write:

**Proposition 4** *Consider closed-loop memoryless equilibria. Under price competition, the amount of effort in TCRD is socially optimal; under quantity competition, the amount of effort in TCRD is socially excessive.*

Now, we proceed with a comparison between market regimes in terms of equilibrium welfare. Steady state welfare levels, gross of investment efforts, are:

$$SW^C = A^2 \frac{3 + \gamma}{(2 + \gamma)^2} \quad (43)$$

$$SW^B = A^2 \frac{3 - 2\gamma}{(1 + \gamma)(2 - \gamma)^2} \quad (44)$$

where superscript  $C$  and  $B$  stand for Cournot and Bertrand, respectively. Note that the state variable does not enter the above welfare expressions, in that it cancels out once equilibrium quantities are plugged into (9). From a direct comparison between (43) and (44) it is straightforward to conclude that  $SW^B > SW^C$  always in the relevant parameter range, since the quantity firms decide to produce under Bertrand competition is always higher than the one they decide to produce under Cournot's, no matter the share of the good that reaches the market. *A fortiori*, taking into account that, as we know from Proposition 3,  $k^{CL} > k^{OL} = k^{BSS}$ , the welfare performance of the Bertrand game is superior to that of the Cournot game, net of steady state investments.

One could ask himself whether the fact that a planner would prefer firms to be price setters depends upon the fact that in this regime market coverage is larger. Reasoning for given quantities produced, meaning that production plans are independent of market regimes, one can investigate upon the effect of TCRD investments on welfare under both kinds of competition. As we have introduced before, (9) is decreasing in the state variable when  $q < As/(1 + \gamma)$ . It can be easily proved that such a condition is always satisfied at equilibrium quantities (22) and (13).

Provided that firms invest more in TCRD under quantity competition, it is now true that  $SW^C > SW^B$ . Net of investment efforts, this inequality may indeed take either sign. A planner willing to ask firms to produce a given amount of substitute goods and ship it to the final market for consumption, is not indifferent over the choice of variable, rather, he might have a strict preference towards quantity competition. The reason lies in the incentives for firms to invest in TCRD, which are higher under Cournot than under Bertrand competition.

## 6 Concluding remarks

An established result states that there is an excess of process-innovating R&D under Cournot competition, while the opposite holds under Bertrand competition (Brander and Spencer, 1983; Dixon, 1985). In this paper we have taken a differential game approach to investments in transport and communication technology, confirming the acquired wisdom. Comparing the closed loop private optima with the social optimum, we have shown that the unique distortion that arises under price competition involves equilibrium prices, above marginal cost due to the presence of market power, while under Cournot competition, together with the usual downward market distortion, the amount of effort in TCRD turns out to be upward distorted. Finally, dealing with a welfare appraisal, we have argued that a planner willing to ask firms to produce a given amount of substitute goods and ship it to the final market for consumption, might have a strict preference towards quantity competition. Once accounted for different production levels, no matter the share of the good that reaches the market, the same planner would always opt for price competition.

# Appendix

## Proof of Proposition 1.

We consider the system composed by (5) in combination with the appropriate kinematics of the control variable  $k_i$ , that is, (17); we form the Jacobian matrix and evaluate it at the steady state:

$$J^B = \left[ \begin{array}{cc} \frac{\partial \dot{s}_i}{\partial s_i} & \frac{\partial \dot{s}_i}{\partial k_i} \\ \frac{\partial \dot{k}_i}{\partial s_i} & \frac{\partial \dot{k}_i}{\partial k_i} \end{array} \right]_{(s_i^{SS}, k_i^{SS})} = \left[ \begin{array}{cc} \rho & -\frac{\alpha\rho}{\delta} \\ -\frac{\delta(\rho + \delta)}{\alpha} & 0 \end{array} \right]$$

Since the determinant of the above  $2 \times 2$  matrix is  $-\rho(\rho + \delta) < 0$ , the equilibrium we have obtained is a saddle. From the phase diagram, it is clear that this saddlepoint equilibrium can be approached only along the north-est arm of the saddle path. ■

## Proof of Proposition 2.

On the basis of the two differential equations system (5) and (30), we form the Jacobian matrix:

$$J^C = \left[ \begin{array}{cc} \frac{\partial \dot{s}_i}{\partial s_i} & \frac{\partial \dot{s}_i}{\partial k_i} \\ \frac{\partial \dot{k}_i}{\partial s_i} & \frac{\partial \dot{k}_i}{\partial k_i} \end{array} \right] = \left[ \begin{array}{cc} -\delta + 2\delta s_i - \alpha k_i & -\alpha(-1 + s_i) \\ -k_i\delta - \frac{A^2\gamma^2\alpha}{4\beta(2 + \gamma)^2 s_i^2} & (\rho + \delta - \delta s_i) \end{array} \right]$$

The saddle path stability of the system requires the negativity of the determinant of the above  $2 \times 2$  matrix evaluated at the steady state point. However, the involved expressions (31) and (32) turn out to be cumbersome. We proceed as follows: we first compute the determinant of the Jacobian matrix:

$$|J^C| = (-\delta + 2\delta s_i - \alpha k_i)(\rho + \delta - \delta s_i) - \alpha(s_i - 1) \left[ k_i\delta + \frac{A^2\gamma^2\alpha}{4\beta(2 + \gamma)^2 s_i^2} \right]$$

Now we define  $\widehat{\beta}$  the value of  $\beta$  such that  $|J^C| = 0$ :

$$\widehat{\beta} = \frac{A^2 \gamma^2 \alpha^2}{(\rho + \delta)(2 + \gamma)^2 - 2s_i^{SS} \delta (4(1 + \gamma) + \gamma^2)}$$

Saddle path stability requires:

$$\begin{aligned} \beta &> \widehat{\beta} \text{ iff } s_i^{CL} > \frac{\rho + \delta}{2\delta}; \\ \beta &< \widehat{\beta} \text{ iff } s_i^{CL} < \frac{\rho + \delta}{2\delta}. \end{aligned}$$

However, the only admissible inequality is  $s_i^{CL} > \frac{\rho + \delta}{2\delta}$ , which, rewritten in terms of  $k_i^{CL}$ , turns out to be  $k_i^{CL} > \frac{\rho + \delta}{2\alpha}$ . From Proposition 3, in fact, we know that  $k^{CL} > k^{OL} = \frac{\rho + \delta}{\alpha}$ . This implies that  $s_i^{CL} < \frac{\rho + \delta}{2\delta}$  is not admissible. Finally, straightforward algebra suffices to prove that, when  $s_i^{SS} > \frac{\rho + \delta}{2\delta}$  then  $\widehat{\beta} < 0$ , meaning that  $\beta > \widehat{\beta}$  always holds. ■

**Value of  $\Psi$ .**

$$\Psi = \left\{ (3A\alpha\beta\gamma)^2 (2 + \gamma)^4 (\rho - 2\delta) - [2\beta (2 + \gamma)^2 (\rho + \delta)]^3 + 3\sqrt{3}\sqrt{\Sigma} \right\}^{\frac{1}{3}}$$

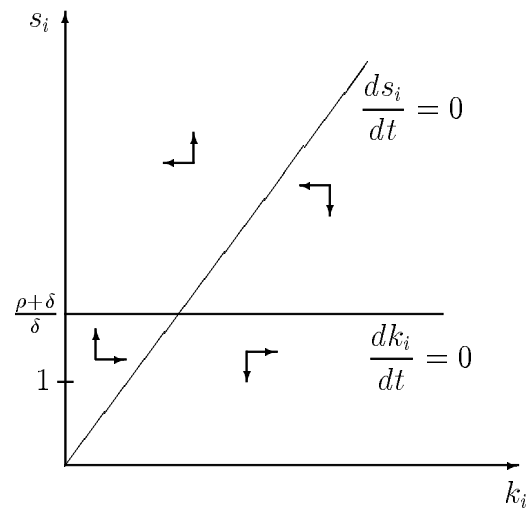
$$\begin{aligned} \Sigma &= \beta [A\alpha\beta\gamma (2 + \gamma)^3]^2 \cdot \\ &\cdot \left[ (A\alpha\gamma)^4 + (4\beta (2 + \gamma)^2)^2 \delta (\rho + \delta)^3 - \beta (A\alpha\gamma (2 + \gamma))^2 (\rho^2 + 20\rho\delta - 8\delta^2) \right] \end{aligned}$$

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# Figures



**Figure 1** : Phase Diagram